Geometric Unity II: The Matter Ledger

Minimal Chiral Completion, One-Family Pati–Salam, Spin(10), Anomaly Closure, and the Axial-Current Handoff

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Abstract

Paper II is the matter, symmetry, and current ledger for the Geometric Unity series. It consumes the classical exports of Paper I—completed curvature, completed torsion, the completed Levi–Civita slice, Projection–Variation, and the sign-safe axial operator basis—and proves the matter data that GU III and GU IV may import without changing conventions.

In a declared global spin, Spin^c, or local tubular EFT domain, the restricted ambient spinor carrier factorizes as

$$S_Y|_X \simeq S_X \otimes_{\mathrm{gr}} S_{\mathrm{int}}.$$

Inside the declared one-family minimal compact completion category, the one-family SM $+\, \nu_R$ chiral input selects the family-universal anomaly-compatible abelian span

$$\mathrm{span}\{Y, B-L\},$$

with $B-L$ the non-hypercharge completion direction. The minimal color–lepton and right-weak completions then give the Pati–Salam carrier

$$W^{\mathrm{LH}}_{\mathrm{fam}} \simeq (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}),$$

and the minimal-rank simple chiral envelope is the **16** of Spin(10). The selection chain used here is the standalone minimal chiral-completion proof record [20] incorporated into the GU II matter ledger.

The family block branches to the Standard Model one-family table with a right-handed neutrino, fixes

$$Y = T^3_R + \frac{1}{2}(B-L),$$

and cancels all perturbative, global, $B-L$, Pati–Salam, Spin(10)-restriction, and six-form anomaly-polynomial obstructions. Paper II also proves the gauge-admissible Yukawa selection rule, identifies the minimal Pati–Salam bi-doublet scalar channel $(\mathbf{1}, \mathbf{2}, \mathbf{2})$, derives the hypercharge kernel and coupling relation, and defines the current/thermal ledger used downstream.

The current ledger has two principal lanes. On a conserved or effectively conserved homogeneous axial branch,

$$n_5(a) = n_{5,0}a^{-3}, \qquad \sigma_0^2 = C_{55}n_{5,0}^2,$$

and fixed-current metric variation gives

$$\rho_5(a) = p_5(a) = -\sigma_0^2 a^{-6}, \qquad w_5 = 1.$$

Paper II also defines the sterile dense-fermion branch as a species-resolved weak-singlet source class with declared polarization, washout, and energy-budget gates. This branch is exported as source-class data for GU III legality checks and downstream species-resolution, source-adapter, and KTC descendant calculations.

Keywords: geometric unity; matter ledger; chimeric spinors; Spin^c factorization; minimal chiral completion; Pati–Salam; Spin(10); one-family block; anomaly cancellation; index polynomial; hypercharge; current ledger; axial current; sterile dense-fermion branch; thermal factors; Yukawa seeds; σ_0 handoff.

Executive Summary

Paper II establishes the matter and current layer of the Geometric Unity series. Paper I supplies the classical geometric spine and the sign-safe axial operator

$$O_{55} = -J_{5\mu} J_5^{\mu}, \qquad \Delta L_X = C_{55} O_{55}, \qquad C_{55} > 0. \tag{0.1}$$

Paper II supplies the spin-domain carrier, minimal chiral-completion chain, one-family matter block, anomaly ledger, Yukawa seed channels, gauge normalization, internal-singlet axial-current bridge, conserved-current handoff, and sterile dense-fermion source-class branch required by downstream GU layers.

The current series records are:

GU I: 10.5281/*zenodo*.17252988, GU II: 10.5281/*zenodo*.17254875,

GU III: 10.5281/*zenodo*.17374258, GU IV: 10.5281/*zenodo*.17374850,

GU V: 10.5281/*zenodo*.17402260.

Scope sentence. Paper II proves a fixed matter/current ledger inside declared spin-domain and one-family minimal compact completion hypotheses. Within that category, the SM + ν_R one-family input selects the Pati–Salam carrier, admits the Spin(10) chiral envelope, identifies the internal-singlet Lorentz axial current, and exports the current branches consumed downstream.

The principal theorem chain is:

E1. Spin-domain carrier. In a declared global spin, Spin^c, or local tubular EFT domain, the restricted ambient spinor carrier factorizes as

$$S_Y|_X \simeq S_X \otimes_{\mathrm{gr}} S_{\mathrm{int}}.$$

The spin-domain theorem supplies S_{int}, the internal carrier on which the matter ledger is built.

E2. Gamma split and chirality. The matrices

$$\Gamma_\mu = \gamma_\mu \otimes \mathbf{1}, \qquad \Gamma_i = \gamma^5 \otimes \kappa_i$$

satisfy the Clifford algebra of $TX \oplus \nu$. The slice chirality projectors P_L, P_R are complementary idempotents and commute with internal gauge actions.

E3. Minimal chiral-completion ladder. The one-family SM + ν_R chiral input, written in all-left-handed notation, has family-universal anomaly-compatible abelian directions

$$X \in \mathrm{span}\{Y, B - L\}.$$

Modulo the observed hypercharge direction, $B - L$ is the anomaly-safe completion direction. The completion is staged: first the minimal complex color–lepton carrier fixes $SU(4)_C$, then the minimal compact right-weak carrier fixes $SU(2)_R$, and finally the minimal-rank simple chiral-envelope problem fixes the Spin(10) envelope.

E4. Pati–Salam family block. Inside the declared one-family minimal compact completion category, the selected family submodule is

$$W^{\mathrm{LH}}_{\mathrm{fam}} \simeq (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}),$$

and its orthogonal projector satisfies

$$[\Pi_{\mathrm{fam}}, \rho_{\mathrm{PS}}(g)] = 0 \qquad \text{for all } g \in G_{\mathrm{PS}}.$$

E5. One-family state table. The family submodule contains exactly sixteen left-handed Weyl states and branches to

$$q_L, \quad \ell_L, \quad u^c_R, \quad d^c_R, \quad e^c_R, \quad \nu^c_R.$$

E6. Hypercharge uniqueness. The linear map

$$Y = aT^3_R + b(B - L)$$

is fixed inside the one-family Pati–Salam table by

$$a = 1, \qquad b = \frac{1}{2}, \qquad Y = T^3_R + \frac{1}{2}(B - L).$$

E7. Spin(10) **envelope.** The minimal-rank simple chiral envelope is Spin(10), with

$$\mathbf{16} \downarrow_{G_{\mathrm{PS}}} = (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}).$$

This is a representation-theoretic envelope for the one-family carrier, not a scalar-potential or mass-spectrum claim.

E8. Anomaly closure. The one-family block cancels all perturbative Standard Model anomalies, the global $SU(2)_L$ Witten anomaly, the full $B - L$ anomaly ledger, the Pati–Salam nonabelian/global checks, and the Spin(10)-restriction check. Equivalently,

$$I_6^{\text{one family}} = 0.$$

E9. Gauge-admissible Yukawa seed channels. A Yukawa seed is gauge-allowed if and only if the participating representation product contains a singlet. For the family block,

$$\overline{\Psi}_L \Psi_R \sim (\mathbf{1}, \mathbf{2}, \mathbf{2}) \oplus (\mathbf{15}, \mathbf{2}, \mathbf{2}),$$

so the minimal scalar channel is the Pati–Salam bi-doublet

$$\Phi_H \sim (\mathbf{1}, \mathbf{2}, \mathbf{2}).$$

E10. Gauge normalization. The normalized $B-L$ generator obeys

$$Q_{BL} = \frac{1}{2}(B-L) = \sqrt{\frac{2}{3}}\,T_{15}, \qquad g_{BL} = \sqrt{\frac{3}{2}}\,g_4.$$

The unbroken hypercharge field is the massless kernel direction of the $SU(2)_R \times U(1)_{B-L}$ neutral mass matrix, and

$$\frac{1}{g_Y^2} = \frac{1}{g_R^2} + \frac{1}{g_{BL}^2}.$$

E11. Internal-singlet axial-current bridge. Minimal ECSK auxiliary torsion couples minimally to the internal-singlet Lorentz axial current,

$$J_5^\mu = \sum_{I=1}^{16} \bar{\psi}_I \gamma^\mu \gamma^5 \psi_I,$$

not to hypercharge, $B-L$, T_R^3, or another internal gauge current.

E12. Current and thermal handoff. A conserved homogeneous axial current satisfies

$$n_5(a) = n_{5,0} a^{-3}.$$

With $C_{55} > 0$, the conserved-current stress tensor gives the negative-stiff branch

$$\rho_5(a) = p_5(a) = -\sigma_0^2 a^{-6}, \qquad \sigma_0^2 = C_{55} n_{5,0}^2, \qquad w_5 = 1.$$

This branch is NEC-violating:

$$\rho_5 + p_5 = -2\sigma_0^2 a^{-6} < 0.$$

Its admissibility as a bounce-supporting or observable cosmological branch is a downstream GU III/GU IV/GU V domain question.

E13. Sterile dense-fermion branch. Paper II also exports the sterile dense-fermion branch as a species-resolved source class:

$$T_{\text{spin}} \ll \bar{\mu}, \qquad \frac{\mu_5}{\bar{\mu}} \gtrsim 0.27, \qquad \Gamma_\chi \tau_{\text{prod}} \lesssim 1, \qquad \rho_{\text{deg}} \lesssim \rho_b.$$

The last two inequalities are branch-admissibility gates. A packet failing the washout gate or the energy gate is outside the sterile dense-fermion branch, even if its formal axial-current algebra is well-defined. Its degenerate polarization satisfies

$$P_A(r) = \frac{r^3 - 1}{r^3 + 1}, \qquad r = \frac{p_{F+}}{p_{F-}},$$

with $P_A \gtrsim 0.68$ at $r \gtrsim 1.74$.

Main Theorem — GU II Matter and Current Export Theorem

Theorem 0.1 (GU II Matter and Current Export Theorem)**.** *Assume the Paper I completed-variable convention ledger, including*

$$\widehat{A} = A - B, \qquad \widehat{F} = F(\widehat{A}) = F(A) - D_A B + B \wedge B, \qquad \widehat{\omega}_B = \omega - \Upsilon_e(B), \qquad T_{\text{aug}} = T(\widehat{\omega}_B),$$

and the sign-safe axial operator basis

$$O_{55} = -J_{5\mu} J_5^{\mu}, \qquad \Delta L_X = C_{55} O_{55}, \qquad C_{55} > 0.$$

Let one of the three spin domains of theorem B.5 be declared: global spin, Spin^c*, or local tubular EFT. Let* S_{int} *be the internal carrier supplied by that domain. Let the one-family* $\mathrm{SM} + \nu_R$ *chiral input be read inside the one-family minimal compact completion category of theorem D.2, and use the all-left-handed anomaly convention of theorem A.7. Then Paper II proves the following exports:*

$$S_Y|_X \simeq S_X \otimes_{\text{gr}} S_{\text{int}}, \tag{0.2}$$

$$X_{\text{abelian}} \in \operatorname{span}\{Y, B - L\}, \tag{0.3}$$

$$W_{\text{fam}}^{\text{LH}} \simeq (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}), \tag{0.4}$$

$$[\Pi_{\text{fam}}, \rho_{\text{PS}}(g)] = 0 \quad \textit{for all } g \in G_{\text{PS}}, \tag{0.5}$$

$$Y = T_R^3 + \frac{1}{2}(B - L), \tag{0.6}$$

$$\mathbf{16}_{\text{Spin}(10)} \downarrow_{G_{\text{PS}}} = (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}), \tag{0.7}$$

$$I_6^{\text{one family}} = 0, \tag{0.8}$$

$$\overline{\Psi}_L \Psi_R \sim (\mathbf{1}, \mathbf{2}, \mathbf{2}) \oplus (\mathbf{15}, \mathbf{2}, \mathbf{2}), \tag{0.9}$$

$$Q_{BL} = \frac{1}{2}(B - L) = \sqrt{\frac{2}{3}} T_{15}, \qquad g_{BL} = \sqrt{\frac{3}{2}} g_4, \tag{0.10}$$

$$\frac{1}{g_Y^2} = \frac{1}{g_R^2} + \frac{1}{g_{BL}^2}, \tag{0.11}$$

$$J_5^{\mu} = \sum_{I=1}^{16} \bar{\psi}_I \gamma^{\mu} \gamma^5 \psi_I \quad \textit{in the minimal internal-singlet torsion channel.} \tag{0.12}$$

If, in addition, the selected homogeneous axial branch satisfies the conserved-branch admissibility condition of theorem 13.1, then

$$n_5(a) = n_{5,0} a^{-3}, \tag{0.13}$$

$$\sigma_0^2 = C_{55} n_{5,0}^2, \tag{0.14}$$

$$\rho_5(a) = p_5(a) = -\sigma_0^2 a^{-6}, \qquad w_5 = 1. \tag{0.15}$$

If the sterile dense-fermion branch is declared, Paper II additionally exports the source-class packet

$$GU2\text{-}K = \left\{ \begin{matrix} \textit{occupied species list}, \quad \textit{weak-singlet/sterile carrier status}, \quad T_{\mathrm{spin}}, \quad \bar{\mu}, \\ \mu_5, \quad P_A(r), \quad \Gamma_\chi \tau_{\mathrm{prod}}, \quad \rho_{\mathrm{deg}}/\rho_b \end{matrix} \right\},$$

with $P_A(r) = (r^3 - 1)/(r^3 + 1)$.

Proof. The spin-domain factorization is theorem B.5. The family-universal abelian direction theorem is theorem D.5. The minimal Pati–Salam carrier and projector compatibility are theorems C.7 and D.11. Hypercharge uniqueness and branching are theorems E.2 and E.4. The minimal-rank simple chiral envelope is theorem 8.2. Anomaly closure is theorems F.6 and F.7. The Yukawa selection rule is theorems G.1 and G.2. Gauge normalization is theorems H.1 and H.2. The internal-singlet axial-current bridge is theorem 12.2. The conserved-branch current export is theorems I.3, I.19, I.20 and 13.1. The sterile dense-fermion branch packet is theorems 13.2 to 13.4 and 13.6. □

Front-Matter Ledgers

Standard physics dictionary

GU/Paper II term	**Standard meaning**
Chimeric carrier	Restricted ambient spinor bundle factored into slice and internal spinor factors.
Spin-domain carrier	Spin, Spin^c, or local tubular domain where the factorization exists.
Minimal chiral completion	Declared one-family, family-universal, spectator-free compact completion category selecting $B-L$, $SU(4)_C$, $SU(2)_R$, and the Pati–Salam carrier.
Family submodule	Gauge-invariant internal Pati–Salam subrepresentation inside S_{int}.
Spin(10) envelope	Minimal-rank simple chiral envelope whose **16** restricts to the one-family Pati–Salam block.
Family projector	Orthogonal projector onto a gauge-invariant internal submodule.
All-left-handed convention	Anomaly convention replacing physical right-handed fields by left-handed conjugates.
Gauge-admissible scalar channel	Scalar representation channel whose product with the fermion bilinear contains a singlet.
Internal-singlet axial current	Lorentz axial current summed over the internal family carrier; the current selected by minimal ECSK torsion.
Current ledger	Definitions and branch conditions for J_5^μ, n_5, $g_{5,\mathrm{eff}}$, conserved/source classes, and σ_0.
Conserved branch	Interval/domain where $\nabla_\mu J_5^\mu = 0$ or effective conservation is justified.
Sterile dense-fermion branch	Species-resolved weak-singlet dense-fermion source class with polarization, washout, and energy-budget gates.

Spin-domain ledger

Every use of the chimeric carrier belongs to exactly one declared domain:

$$\text{Global spin domain:} \quad S_Y|_X \simeq S_X \otimes_{\mathrm{gr}} S_\nu, \tag{0.16}$$

$$\text{Spin}^c \text{ domain:} \quad S_Y^c|_X \simeq S_X^c \otimes_{\mathrm{gr}} S_\nu^c, \tag{0.17}$$

$$\text{Local tubular EFT domain:} \quad S_Y|_{U\cap X} \simeq S_X|_{U\cap X} \otimes_{\mathrm{gr}} S_{\mathrm{int}}|_{U\cap X}. \tag{0.18}$$

Claim-status table

Claim	Status
Spin/Spin^c/local factorization	Theorem under domain hypotheses.
Gamma split	Theorem.
Chirality/projector compatibility	Theorem.
Family-universal abelian direction	Theorem inside the one-family minimal compact completion category.
Pati–Salam carrier	Theorem inside the one-family minimal compact completion category.
Family projector	Theorem.
One-family state table	Theorem.
Hypercharge map	Theorem under the state-table requirements.
Spin(10) envelope	Theorem inside the minimal-rank simple chiral-envelope category.
Anomaly closure	Theorem.
Yukawa seed selection	Theorem.
Internal-singlet axial current	Theorem for minimal ECSK torsion coupling.
Geometry-sourced scalar existence	Separate internal-module task.
Conserved axial branch	Conditional domain.
σ_0 handoff	Theorem on conserved/effectively conserved branch.
Sterile dense-fermion branch	Source-class export with GU III legality required for downstream activation.
Numerical fermion masses	Downstream.
Observational fits	Downstream.
Three-family replication	Separate-domain task.

Three quick diagrams

$$Y \supset X \quad \Longrightarrow \quad S_Y|_X \quad \Longrightarrow \quad S_X \otimes_{\mathrm{gr}} S_{\mathrm{int}} \quad \Longrightarrow \quad W_{\mathrm{fam}}^{\mathrm{LH}}.$$

Carrier pipeline. The spin-domain theorem supplies S_{int}; the one-family minimal compact completion theorem supplies $W_{\mathrm{fam}}^{\mathrm{LH}}$.

Figure 1: Carrier pipeline from two-space geometry to the one-family internal submodule.

$$\mathrm{SM} + \nu_R \Longrightarrow \mathrm{span}\{Y, B-L\} \Longrightarrow SU(4)_C \times SU(2)_L \times SU(2)_R \Longrightarrow \mathbf{16}_{\mathrm{Spin}(10)}.$$

Minimal chiral-completion ladder. The one-family input selects the Pati–Salam carrier inside the declared category and admits the minimal-rank simple chiral envelope.

Figure 2: Minimal chiral-completion ladder for the one-family matter carrier.

$$(\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}) \quad \Longrightarrow \quad q_L \oplus \ell_L \oplus u_R^c \oplus d_R^c \oplus e_R^c \oplus \nu_R^c.$$

Branching tree. The all-left-handed table is the anomaly-ready branch of the Pati–Salam family block.

Figure 3: Pati–Salam branching into the all-left-handed one-family state table.

$$\text{state table} \Longrightarrow \text{anomaly closure} \Longrightarrow J_5^\mu \Longrightarrow \sigma_0^2 = C_{55} n_{5,0}^2 \Longrightarrow \text{GU IV packet.}$$

$$\text{sterile dense-fermion branch} \Longrightarrow \text{GU2-K} \Longrightarrow \text{GU III legality} \Longrightarrow \text{KT0/KTC source packet.}$$

Ledger-to-observable handoff. GU IV may use the conserved branch only on a declared conserved/effectively conserved interval; the sterile dense-fermion branch enters downstream through legality and species-resolution gates.

Figure 4: Matter ledger to current ledger to conserved and source-class downstream packets.

Central one-family state table

Physical	**LH anomaly field**	$SU(3)_c$	$SU(2)_L$	T_R^3	$B-L$	Y	**Mult.**
q_L	$q_L = (u_L, d_L)$	$\mathbf{3}$	$\mathbf{2}$	0	1/3	1/6	6
ℓ_L	$\ell_L = (\nu_L, e_L)$	$\mathbf{1}$	$\mathbf{2}$	0	−1	−1/2	2
u_R	u_R^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	−1/2	−1/3	−2/3	3
d_R	d_R^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	+1/2	−1/3	+1/3	3
e_R	e_R^c	$\mathbf{1}$	$\mathbf{1}$	+1/2	+1	+1	1
ν_R	ν_R^c	$\mathbf{1}$	$\mathbf{1}$	−1/2	+1	0	1

$$6 + 2 + 3 + 3 + 1 + 1 = 16.$$

The charges in the $u_R^c, d_R^c, e_R^c, \nu_R^c$ rows are the charges of the left-handed conjugate fields, not the physical right-handed fields.

Scope Note 0.2 (One family versus three families)**.** Paper II proves the minimal one-family matter packet. Family replication is a forward selection problem. Direct replication, index multiplicity, discrete geometry, and internal spectral degeneracy are possible downstream mechanisms; each requires its own theorem ledger.

Proof Walk-Through for Technical Readers

A technical reviewer can audit the paper by following the appendix spine below. Each main-text export is backed by a theorem, proposition, lemma, definition, export, domain condition, scope boundary, or review criterion in the appendices.

Table 1: Fast proof walk-through.

Step	Claim	What is proved	Where
1	Convention lock	Completed-curvature sign, axial sign basis, all-left-handed anomaly convention, internal-singlet axial-current convention, and trace rules.	Section A
2	Chimeric carrier	Factorization is valid in exactly the declared global spin, Spin^c, or local tubular EFT domains.	theorem B.5
3	Gamma split	$\Gamma_\mu = \gamma_\mu \otimes \mathbf{1}$, $\Gamma_i = \gamma^5 \otimes \kappa_i$ satisfy the Clifford algebra.	theorem C.1
4	Projectors	Slice chirality and internal/family projectors commute under the stated unitary hypotheses.	theorems C.3 and C.6
5	Abelian direction	Family-universal anomaly-compatible abelian charges lie in $\mathrm{span}\{Y, B-L\}$.	theorem D.5
6	Pati–Salam carrier	The one-family minimal compact completion category selects $W^{\mathrm{LH}}_{\mathrm{fam}}$.	theorem D.11
7	Family projector	$[\Pi_{\mathrm{fam}}, \rho_{\mathrm{PS}}(g)] = 0$.	theorem C.7
8	State table	Sixteen left-handed Weyl states and the displayed table.	theorems D.13 and D.15
9	Hypercharge	$Y = T^3_R + (B-L)/2$ is fixed by the table requirements.	theorem E.2
10	Spin(10) envelope	The **16** restricts to the spectator-free Pati–Salam family block.	theorem 8.2
11	Anomalies	Standard Model, $B-L$, Pati–Salam, Spin(10)-restriction, Witten $SU(2)$, and I_6 checks close.	theorems F.1, F.6 and F.7
12	Yukawa seeds	Scalar channels are gauge-admissible iff a singlet exists.	theorems G.1 and G.2
13	Gauge normalization	g_{BL}, hypercharge kernel, and g_Y relation.	theorems H.1 and H.2
14	Axial-current bridge	Minimal ECSK torsion selects the internal-singlet Lorentz axial current.	theorem 12.2

Step	Claim	What is proved	Where
15	Current handoff	Conserved branch $n_5 \propto a^{-3}$; current stress tensor gives $\rho_5 = p_5 = -\sigma_0^2 a^{-6}$.	theorems I.3 and I.19
16	Sterile dense-fermion branch	Species-resolved branch gates, degenerate polarization $P_A(r)$, energy-budget gate, and washout gate.	theorems I.8, I.11, I.13 and I.15
17	Reproducibility	Artifact status requires manifest rows with filename, version, date, checksum, and proof label.	theorem J.4

Reviewer shortcut. For the mathematical spine, read Sections A to F and I. For the downstream export protocol, read Section J.

Contents

1 Introduction

1.1 Why Paper II matters

Paper I fixes the completed geometry. Paper II fixes the matter ledger that may live on that geometry. This division is essential: once the completed curvature, completed torsion, Projection–Variation rule, and axial operator basis are fixed, downstream papers still need to know which spinor carrier is being used, which internal family block is allowed, which charge table is legal, which anomalies cancel, and which axial current is being handed forward.

Paper II supplies that contract. It prevents later layers from choosing matter content, hypercharge conventions, anomaly sums, current normalizations, or thermal branches after the fact. The one-family block fixes the charge table. The anomaly ledger certifies the matter packet. The current ledger specifies which axial-current branches GU III may certify and GU IV may map into observables. The result is a theorem-grade matter-import layer, not a phenomenological model fit.

The guiding chain is

$$\text{completed geometry} \Longrightarrow \text{spinor carrier} \Longrightarrow \text{one-family matter block},$$
$$\text{one-family matter block} \Longrightarrow \text{anomaly-safe current ledger} \Longrightarrow \text{downstream observable packet}.$$

The purpose of Paper II is to make every arrow in that chain explicit.

1.2 Purpose of Paper II

Geometric Unity is organized as a sequence of proof layers. Paper I establishes the classical geometric layer: completed curvature, completed torsion, the completed Levi–Civita slice, Projection–Variation, and the sign-safe axial operator basis. Paper II establishes the matter and current layer. Its task is to answer a precise question:

> *Given the completed slice geometry and axial operator exported by Paper I, what spinor carrier, internal family block, anomaly ledger, current normalization, and downstream handoff may GU III and GU IV safely consume?*

The answer is the Paper II matter packet. In a declared spin domain, the restricted ambient spinor bundle factorizes into a slice spinor factor and an internal factor,

$$S_Y|_X \simeq S_X \otimes_{\mathrm{gr}} S_{\mathrm{int}}. \tag{1.1}$$

This factorization supplies the internal carrier S_{int}. The carrier is then read inside the one-family minimal compact completion category developed in the standalone minimal chiral-completion proof record [20]. In that category, the one-family $\mathrm{SM} + \nu_R$ chiral input fixes the family-universal anomaly-compatible abelian span

$$X_{\mathrm{abelian}} \in \mathrm{span}\{Y, B - L\}. \tag{1.2}$$

Modulo the observed hypercharge direction, $B - L$ is the anomaly-safe completion direction. The minimal color–lepton and right-weak completions then give the Pati–Salam family carrier

$$W_{\text{fam}}^{\text{LH}} \simeq (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}). \tag{1.3}$$

The family projector is gauge-compatible:

$$[\Pi_{\text{fam}}, \rho_{\text{PS}}(g)] = 0 \qquad \text{for all } g \in G_{\text{PS}}. \tag{1.4}$$

The same carrier admits a minimal-rank simple chiral envelope:

$$\mathbf{16}_{\text{Spin}(10)} \downarrow_{G_{\text{PS}}} = (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}). \tag{1.5}$$

This envelope is a representation-theoretic bookkeeping result. It supplies a clean simple-envelope reference for GU III anomaly and BRST/BV bookkeeping. It is not a scalar-potential claim, not a mass-spectrum claim, and not a low-energy unification assertion.

Domain Condition 1.1 (Minimal chiral-completion domain)**.** The implication proved in Paper II is

$$\text{spin domain} \Longrightarrow S_{\text{int}},$$

followed by the one-family minimal compact completion chain

$$\text{SM} + \nu_R \Longrightarrow \text{span}\{Y, B - L\} \Longrightarrow SU(4)_C \times SU(2)_L \times SU(2)_R \Longrightarrow W_{\text{fam}}^{\text{LH}} \Longrightarrow \mathbf{16}_{\text{Spin}(10)}.$$

The conclusion is a category theorem inside the declared one-family, family-universal, spectator-free, minimal compact completion problem. Nonminimal extensions, spectator-containing models, alternative internal geometries, and multi-family constructions require their own ledgers.

1.3 Inputs from Paper I

Paper II consumes the following classical inputs from Paper I:

$$\widehat{A} = A - B, \qquad \widehat{F} = F(\widehat{A}) = F(A) - D_A B + B \wedge B, \tag{1.6}$$

$$\widehat{\omega}_B = \omega - \Upsilon_e(B), \qquad T_{\text{aug}} = T(\widehat{\omega}_B), \tag{1.7}$$

$$O_{55} = -J_{5\mu} J_5^{\mu}, \qquad \Delta L_X = C_{55} O_{55}, \qquad C_{55} > 0. \tag{1.8}$$

The sign in (1.6) follows from $\widehat{A} = A - B$ and is proved directly in theorem A.1. The axial basis is fixed with the same level of rigidity. In mostly-plus signature, a homogeneous timelike axial current

$$J_5^{\mu} = n_5 u^{\mu}, \qquad u_\mu u^\mu = -1,$$

satisfies

$$J_{5\mu}J_5^\mu = -n_5^2, \qquad O_{55} = n_5^2 > 0, \tag{1.9}$$

as proved in theorem A.4.

These imported equations are read-only in Paper II. The matter ledger may identify the current that enters O_{55}, and it may classify the branches on which n_5 evolves. It does not alter the completed curvature, completed torsion, axial operator basis, or $C_{55} > 0$ sign convention fixed by Paper I.

1.4 Why the all-left-handed convention matters

Anomaly computations are performed over left-handed Weyl fermions. Therefore physical right-handed fields must be converted to left-handed conjugates:

$$f_R \rightsquigarrow f_R^c. \tag{1.10}$$

For example,

$$u_R : (\mathbf{3}, \mathbf{1})_{2/3} \quad \rightsquigarrow \quad u_R^c : (\bar{\mathbf{3}}, \mathbf{1})_{-2/3}. \tag{1.11}$$

This conversion is not cosmetic. It determines the sign of abelian charges and the conjugation of complex nonabelian representations inside the anomaly sums.

In physical chiral notation, the Pati–Salam one-family block is

$$(\mathbf{4}, \mathbf{2}, \mathbf{1})_L \oplus (\mathbf{4}, \mathbf{1}, \mathbf{2})_R.$$

In the all-left-handed anomaly ledger it becomes

$$(\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}).$$

The bar on the second $SU(4)$ factor is therefore part of the anomaly bookkeeping. It is the reason the $SU(4)^3$, Standard Model, $B - L$, and six-form anomaly ledgers close in the form used by GU III.

1.5 Matter carrier, axial current, and σ_0 handoff

The current ledger is the bridge from the matter table to the observable packet in GU IV. Minimal Einstein–Cartan torsion couples to the Lorentz axial current obtained by tracing over the internal family carrier:

$$J_5^\mu = \sum_{I=1}^{16} \bar{\psi}_I \gamma^\mu \gamma^5 \psi_I. \tag{1.12}$$

This is the internal-singlet axial current. It is not a hypercharge current, not a $B - L$ current, and not a T_R^3 current. The internal generators act on S_{int}; minimal torsion acts on the Lorentz spin connection. The current selected by the minimal torsion channel is therefore the Lorentz axial singlet over the internal carrier.

The axial density measured by a comoving observer is

$$n_5 = -u_\mu J_5^\mu.$$

If the selected axial current belongs to a conserved branch,

$$\nabla_\mu J_5^\mu = 0,$$

then in FLRW,

$$n_5(a) = n_{5,0} a^{-3}.$$

Combining this with the Paper I axial contact coefficient $C_{55} > 0$ gives

$$\sigma_0^2 = C_{55} n_{5,0}^2.$$

The conserved-current stress tensor then yields

$$\rho_5(a) = p_5(a) = -\sigma_0^2 a^{-6}, \qquad w_5 = 1.$$

If the axial current is anomalous or source-driven, GU IV must use the sourced evolution equation rather than the conserved a^{-3} law.

1.6 The sterile dense-fermion branch

The conserved homogeneous branch is one current lane. Paper II also records a species-resolved source-class lane: the sterile dense-fermion branch. This branch is designed for a weak-singlet or otherwise sphaleron-blind dense Dirac sector whose axial occupation can survive over the relevant production interval.

The branch is defined by four gates:

$$T_{\rm spin} \ll \bar{\mu}, \qquad \frac{\mu_5}{\bar{\mu}} \gtrsim 0.27, \qquad \Gamma_\chi \tau_{\rm prod} \lesssim 1, \qquad \rho_{\rm deg} \lesssim \rho_b.$$

The first two gates state that the source is a degenerate spin-current branch with a sufficient axial split. The third gate is the washout condition. The fourth gate is the energy-budget condition.

For a degenerate branch,

$$n_\pm = \frac{g_p}{6\pi^2} p_{F\pm}^3, \qquad P_A = \frac{n_+ - n_-}{n_+ + n_-}.$$

Therefore

$$P_A(r) = \frac{r^3 - 1}{r^3 + 1}, \qquad r = \frac{p_{F+}}{p_{F-}}.$$

The threshold

$$P_A \gtrsim 0.68$$

is reached at

$$r \gtrsim 1.74.$$

This is a matter/current population theorem. GU II defines the branch and its gates. GU III certifies anomaly, washout, boundary, and sign-corridor legality. KT0 and KTC source-resolution layers compute the species-resolved parent-collapse packet. KTC II computes tensor-amplitude descendants. GU IV maps declared source-adapter packets to observables.

1.7 Scope and domain boundaries

Paper II proves controlled chimeric spin factorization domains; the one-family minimal chiral-completion carrier; the sixteen-state all-left-handed table; hypercharge; anomaly closure; gauge-admissible Yukawa seed channels; gauge normalization; the internal-singlet axial-current bridge; conserved and sourced current ledgers; and the sterile dense-fermion source-class branch.

The forward tasks are separated by layer. Numerical fermion masses, scalar-potential stability, vacuum selection, heavy gauge-boson spectra, BRST/BV quantization, loop running, family replication, species-resolved parent-collapse packets, detector amplitudes, and observational fits each require downstream or separate-domain theorem data. This separation is the point of the ledger: GU III certifies legality, GU IV builds observable maps, GU V audits declared packets, and KTC source-resolution layers compute parent-collapse and descendant packets without redefining the GU II matter carrier.

1.8 Roadmap

Section 2 records the imported Paper I conventions. Section 3 states the spin-domain carrier. Sections 5 to 8 give the minimal chiral-completion ladder, family block, state table, branching, hypercharge, and Spin(10) envelope. Section 9 gives the anomaly ledger. Sections 10 to 13 give scalar channels, gauge normalization, the internal-singlet axial-current bridge, conserved-current handoff, and sterile dense-fermion branch. Sections 14 and 15 give the downstream handoff and public-facing domain boundaries.

2 Series Position and Inputs from Paper I

Paper II proves the matter and current ledger used by the downstream Geometric Unity papers. It consumes Paper I through named classical exports and then proves the spinor-carrier, family-block, anomaly, Yukawa, gauge-normalization, axial-current, and source-class data required by GU III and GU IV.

The matter ledger has two logically separate jobs. First, it identifies the admissible internal carrier and the one-family chiral block that may be placed on the completed slice geometry of Paper I. Second, it identifies the Lorentz axial current supplied by that carrier and records the current-branch conditions under which the axial contact exported by Paper I may be consumed downstream.

Proposition 2.1 (Paper II input contract). *Paper II consumes the completed-variable and axial-sign conventions* (1.6)–(1.8) *without modification. Every matter-layer theorem in this paper is a theorem about spinor carriers, internal representations, anomaly ledgers, Yukawa seed channels, gauge normalizations, and currents built on top of the completed Paper I geometry.*

Proof. The completed-curvature identity is proved in theorem A.1. The timelike axial-current sign is proved in theorem A.4. The convention ledger is exported in theorem A.13. These results fix the geometric and sign inputs before the matter layer is constructed. □

Convention Condition 2.2 (Axial-basis consistency). Every occurrence of the axial contact in Paper II is written in the sign-safe basis

$$O_{55} = -J_5^2.$$

If a raw J_5^2 contraction appears, the same line displays

$$J_{5\mu}J_5^{\mu} = -n_5^2, \qquad O_{55} = n_5^2.$$

Scope Boundary 2.3 (Paper II scope). Paper II supplies the matter/current packet. It does not compute a species-resolved parent-collapse packet, a tensor amplitude, a detector-facing gravitational-wave floor, or a cosmological likelihood. Those are downstream KT0/KTC, GU IV, and GU V tasks. Paper II also does not certify the full quantum legality of every current branch; GU III supplies the BRST/BV, anomaly-import, washout, boundary, RG/matching, and sign-corridor certificates.

The minimal chiral-completion and axial-current selection chain used in the matter ledger is developed in the standalone proof record [20]; the present paper incorporates that chain into the GU II downstream export ledger.

3 Spin Domains, Gamma Split, and the Chimeric Carrier

Paper II uses exactly three spin-domain carrier statements:

1. a global spin domain;
2. a Spin^c domain with determinant-line data;
3. a local tubular EFT domain.

The domain-control theorem is theorem B.5.

Theorem 3.1 (Chimeric carrier on the observation slice)**.** *In any declared spin domain of theorem B.5, the restricted ambient spinor carrier factorizes as*

$$S_Y|_X \simeq S_X \otimes_{\mathrm{gr}} S_{\mathrm{int}}, \tag{3.1}$$

or, in the Spin^c *case, as*

$$S^c_Y|_X \simeq S^c_X \otimes_{\mathrm{gr}} S^c_{\mathrm{int}}. \tag{3.2}$$

Proof. The global spin factorization is theorem B.2. The Spin^c factorization is theorem B.3. The local tubular factorization is theorem B.4. The domain theorem theorem B.5 states that every use of (3.1) belongs to one of these domains. □

Once a spin domain is declared, Paper II proves the gamma split

$$\Gamma_\mu = \gamma_\mu \otimes \mathbf{1}, \qquad \Gamma_i = \gamma^5 \otimes \kappa_i. \tag{3.3}$$

The matrices in (3.3) satisfy the Clifford algebra of $TX \oplus \nu$, as proved in theorem C.1. The slice chirality projectors are

$$P_L = \frac{1-\gamma^5}{2}, \qquad P_R = \frac{1+\gamma^5}{2}. \tag{3.4}$$

They are complementary idempotents and commute with internal gauge actions.

Proposition 3.2 (Independence of slice chirality and internal labels)**.** *Slice chirality and internal representation labels are independent pieces of the chimeric tensor-product bookkeeping. Internal gauge transformations do not mix the P_L-image with the P_R-image.*

Proof. This is theorem C.5, which follows from the commutators in theorem C.4. □

Scope Boundary 3.3 (Carrier versus family representation). The chimeric factorization supplies an internal carrier S_{int}. It does not by itself select a Standard Model family representation, a Pati–Salam block, a Spin(10) envelope, or a current branch. Those selections are representation-theoretic matter-ledger results established below.

4 Minimal Chiral Completion Ladder

The internal carrier supplied by the spin-domain theorem must be populated by a chiral matter block. Paper II treats this as a category problem: the observed one-family $\mathrm{SM} + \nu_R$ spectrum is the input, and the admissible completion is required to be family-universal, spectator-free, and compatible with the all-left-handed anomaly convention.

Definition 4.1 (One-family minimal chiral-completion category)**.** The one-family minimal chiral-completion category consists of internal completions satisfying the following requirements:

(i) The input carrier is the one-family Standard Model chiral spectrum with a right-handed neutrino, written entirely as left-handed Weyl fields:

$$q_L, \quad \ell_L, \quad u_R^c, \quad d_R^c, \quad e_R^c, \quad \nu_R^c.$$

(ii) Any new abelian direction is family-universal and commutes with $SU(3)_c \times SU(2)_L$.

(iii) The nonabelian completion is spectator-free: under restriction to the observed subgroup it contains exactly the one-family carrier and no additional irreducible matter components.

(iv) The color–lepton completion acts on a complex irreducible carrier whose restriction to $SU(3)_c \times U(1)_{B-L}$ contains exactly

$$\mathbf{4} \downarrow_{SU(3)_c \times U(1)_{B-L}} = (\mathbf{3}, 1/3) \oplus (\mathbf{1}, -1),$$

with no additional summands.

(v) The right-weak completion acts irreducibly on the two-state pairs determined by

$$T_R^3 := Y - \frac{1}{2}(B - L).$$

(vi) Minimality is imposed first on the compact simple color–lepton and right-weak completions, and then on the rank of a compact connected simple chiral envelope containing the Pati–Salam group with a spectator-free irreducible complex representation restricting to the one-family Pati–Salam carrier.

Theorem 4.2 (Family-universal abelian directions)**.** *For the one-family* $\mathrm{SM} + \nu_R$ *carrier in the all-left-handed convention, every family-universal anomaly-compatible abelian charge commuting with* $SU(3)_c \times SU(2)_L$ *lies in*

$$\mathrm{span}\{Y, B - L\}.$$

Consequently, modulo the observed hypercharge direction, $B - L$ *is the unique new family-universal anomaly-free abelian direction in this category.*

Proof. Let a candidate family-universal abelian charge assign values

$$X = (x_q, x_\ell, x_u, x_d, x_e, x_\nu)$$

to

$$q_L, \quad \ell_L, \quad u_R^c, \quad d_R^c, \quad e_R^c, \quad \nu_R^c.$$

The $SU(3)_c^2 U(1)_X$ anomaly gives

$$2x_q + x_u + x_d = 0,$$

and the $SU(2)_L^2 U(1)_X$ anomaly gives

$$3x_q + x_\ell = 0.$$

Hence

$$x_\ell = -3x_q, \qquad x_d = -2x_q - x_u.$$

The mixed gravitational anomaly gives

$$6x_q + 2x_\ell + 3x_u + 3x_d + x_e + x_\nu = 0.$$

Substituting the two nonabelian relations gives

$$6x_q + 2(-3x_q) + 3x_u + 3(-2x_q - x_u) + x_e + x_\nu = 0,$$

so

$$x_e + x_\nu = 6x_q.$$

Because hypercharge is retained as a gauge direction, anomaly compatibility of the added abelian direction also requires the mixed abelian coefficient $U(1)_Y^2 U(1)_X$ to vanish. Using the all-left-handed hypercharges, this condition is

$$6\left(\frac{1}{6}\right)^2 x_q + 2\left(-\frac{1}{2}\right)^2 x_\ell + 3\left(-\frac{2}{3}\right)^2 x_u + 3\left(\frac{1}{3}\right)^2 x_d + x_e = 0.$$

Substituting $x_\ell = -3x_q$ and $x_d = -2x_q - x_u$ gives

$$-2x_q + x_u + x_e = 0,$$

hence

$$x_e = 2x_q - x_u, \qquad x_\nu = 4x_q + x_u.$$

Thus every compatible charge vector has the form

$$X = (x_q,\, -3x_q,\, x_u,\, -2x_q - x_u,\, 2x_q - x_u,\, 4x_q + x_u)\,.$$

Solving

$$X = aY + b(B - L)$$

from the q_L and u_R^c entries gives

$$a = -2x_q - 2x_u, \qquad b = 4x_q + x_u.$$

Substitution into $aY + b(B - L)$ reproduces all six entries above. Therefore

$$X \in \operatorname{span}\{Y, B - L\}.$$

The remaining cubic abelian coefficient and the $U(1)_Y U(1)_X^2$ coefficient vanish on this two-parameter span because the one-family SM $+\,\nu_R$ table is anomaly-free for both Y and $B - L$, including the mixed abelian coefficients. Since Y is already the observed Standard Model hypercharge, the independent new anomaly-free direction is $B - L$, up to normalization and addition of Y. □

Lemma 4.3 (Minimal color–lepton completion)**.** *Inside the category of theorem 4.1, the minimal compact simple color–lepton completion of*

$$(\mathbf{3}, 1/3) \oplus (\mathbf{1}, -1)$$

is $SU(4)_C$, with fundamental branching

$$\mathbf{4} \downarrow_{SU(3)_c \times U(1)_{B-L}} = (\mathbf{3}, 1/3) \oplus (\mathbf{1}, -1).$$

Proof. The desired carrier has complex dimension four and contains a color triplet plus a lepton singlet with the traceless charge pattern proportional to

$$\operatorname{diag}(1, 1, 1, -3).$$

The relevant smaller and competing simple compact possibilities are excluded as follows:

Candidate	**Small carrier**	**Obstruction**
$SU(2)$	$\mathbf{2}$	No $SU(3)_c$ subgroup and no color triplet.
$SU(3)$	$\mathbf{3}, \bar{\mathbf{3}}$	No independent lepton singlet inside the irreducible carrier.
$\mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$	$\mathbf{4}$ pseudoreal	Does not restrict as $(\mathbf{3})_{1/3} \oplus (\mathbf{1})_{-1}$.
G_2	$\mathbf{7}$ real	Not a minimal complex four-state color–lepton carrier.

A carrier containing $SU(3)_c \times U(1)_{B-L}$ with the required triplet-plus-singlet branching must have rank at least three. Among rank-three simple compact groups, the fundamental of $SU(4)$ realizes the required complex four-state branch:

$$\mathbf{4} \to (\mathbf{3})_{1/3} \oplus (\mathbf{1})_{-1}.$$

The competing rank-three families do not supply a complex four-dimensional irreducible carrier with this restriction: Sp(3) has pseudoreal fundamental of complex dimension six, and Spin(7) has real vector and spinor carriers of dimensions seven and eight. Hence the minimal color–lepton completion in the declared category is $SU(4)_C$. □

Lemma 4.4 (Minimal right-weak completion)**.** *With*

$$T_R^3 = Y - \frac{1}{2}(B - L),$$

the right-conjugate states form two-state weight systems with weights $\pm\frac{1}{2}$*. The minimal compact simple right-weak completion connecting those weights is* $SU(2)_R$*.*

Proof. For the right-conjugate quark pair,

$$T_R^3(u_R^c) = -\frac{2}{3} - \frac{1}{2}\left(-\frac{1}{3}\right) = -\frac{1}{2},$$

and

$$T_R^3(d_R^c) = \frac{1}{3} - \frac{1}{2}\left(-\frac{1}{3}\right) = \frac{1}{2}.$$

For the right-conjugate lepton pair,

$$T_R^3(\nu_R^c) = 0 - \frac{1}{2}(1) = -\frac{1}{2}, \qquad T_R^3(e_R^c) = 1 - \frac{1}{2}(1) = \frac{1}{2}.$$

A single $U(1)$ can assign these weights but cannot irreducibly connect the two states. The minimal compact simple Lie algebra with a two-dimensional irreducible representation carrying weights $\pm\frac{1}{2}$ is $\mathfrak{su}(2)$, with raising and lowering generators satisfying

$$[T_R^3, T_R^\pm] = \pm T_R^\pm, \qquad [T_R^+, T_R^-] = 2T_R^3.$$

The left sector has $T_R^3 = 0$ and is therefore an $SU(2)_R$ singlet. Thus the minimal right-weak completion is $SU(2)_R$. □

Theorem 4.5 (Minimal chiral completion ladder)**.** *Within the category of theorem 4.1, the one-family* $\mathrm{SM} + \nu_R$ *carrier selects*

$$B - L, \qquad SU(4)_C, \qquad SU(2)_R, \qquad G_{\mathrm{PS}} = SU(4)_C \times SU(2)_L \times SU(2)_R.$$

The resulting one-family all-left-handed carrier is

$$W_{\mathrm{fam}}^{\mathrm{LH}} \simeq (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}).$$

Proof. The abelian direction theorem theorem 4.2 identifies $B-L$ as the unique new family-universal anomaly-free abelian direction modulo hypercharge. The minimal color–lepton completion lemma theorem 4.3 promotes $SU(3)_c \times U(1)_{B-L}$ to $SU(4)_C$ on the complex four-state color–lepton carrier. The right-weak completion lemma theorem 4.4 promotes the right-conjugate two-state weight systems to $SU(2)_R$. Combining these with the existing weak factor $SU(2)_L$ gives

$$G_{\mathrm{PS}} = SU(4)_C \times SU(2)_L \times SU(2)_R.$$

The left weak quark and lepton doublets assemble as $(\mathbf{4}, \mathbf{2}, \mathbf{1})$. The right-conjugate quark and lepton states assemble as $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$. Therefore the one-family carrier is $W^{\mathrm{LH}}_{\mathrm{fam}}$. □

Scope Boundary 4.6 (Selection domain)*.* The selection theorem is a theorem inside the declared one-family minimal chiral-completion category. It is not a claim that every logically possible internal geometry, family-replication mechanism, scalar sector, or high-energy gauge model is forced to be Pati–Salam. It states that the observed one-family SM $+\,\nu_R$ carrier selects the Pati–Salam family block under the stated family-universal, spectator-free, minimal compact-completion hypotheses.

5 Pati–Salam Admissibility and Gauge-Compatible One-Family Realization

The selection ladder in this section follows the standalone minimal chiral-completion proof record [20]: the one-family SM + ν_R spectrum fixes the family-universal abelian direction modulo hypercharge, promotes $B - L$ to the $SU(4)_C$ color–lepton carrier, promotes the right-sector charge to $SU(2)_R$, and embeds the resulting Pati–Salam family block into the minimal-rank simple chiral envelope Spin(10).

The spin-domain theorem supplies the carrier S_{int}. The minimal chiral-completion ladder supplies the one-family target block $W^{\text{LH}}_{\text{fam}}$. The remaining representation-theoretic statement is the gauge-compatible placement of that target block inside the internal carrier.

Let

$$G_{\text{PS}} = SU(4)_C \times SU(2)_L \times SU(2)_R \tag{5.1}$$

and let

$$\rho_{\text{PS}} : G_{\text{PS}} \to U(S_{\text{int}}) \tag{5.2}$$

be a unitary internal action. A one-family minimal Pati–Salam realization is a G_{PS}-submodule $W_{\text{fam}} \subset S_{\text{int}}$ satisfying the admissibility axioms in theorem D.4: unified color–lepton carrier, right-handed neutrino, left and right weak chiral carriers before breaking, linear hypercharge, anomaly-safety, and no spectator states.

Theorem 5.1 (Gauge-compatible one-family submodule)**.** *If S_{int} carries the one-family minimal chiral-completion realization selected in theorem 4.5, then the all-left-handed family submodule is*

$$W^{\text{LH}}_{\text{fam}} \simeq (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}), \tag{5.3}$$

inside the declared one-family minimal Pati–Salam realization, up to charge conjugation and family replication. If Π_{fam} is the orthogonal projector onto W_{fam}, then

$$[\Pi_{\text{fam}}, \rho_{\text{PS}}(g)] = 0 \qquad \textit{for all } g \in G_{\text{PS}}. \tag{5.4}$$

Proof. The family submodule is the representation selected by theorem 4.5. Since ρ_{PS} is unitary, any invariant finite-dimensional submodule has an invariant orthogonal complement. Therefore the orthogonal projector onto W_{fam} commutes with the G_{PS} action:

$$[\Pi_{\text{fam}}, \rho_{\text{PS}}(g)] = 0.$$

Equivalently, this is the internal-projector compatibility theorem theorem C.6 applied to the family submodule. □

Scope Note 5.2 (Admissibility versus global uniqueness)**.** The family submodule is selected inside the declared one-family minimal chiral-completion category and then realized as a gauge-compatible submodule of S_{int}. The statement does not assert global uniqueness among all possible internal geometries, family-replication mechanisms, or scalar sectors.

6 One-Family State Table and All-Left-Handed Convention

Paper II proves the one-family state table from the family submodule. The submodule contains exactly sixteen left-handed Weyl states:

$$\dim(\mathbf{4},\mathbf{2},\mathbf{1}) = 8, \qquad \dim(\bar{\mathbf{4}},\mathbf{1},\mathbf{2}) = 8, \qquad 8+8=16. \tag{6.1}$$

This count is proved in theorems D.13 and D.15 and matches the one-family SM $+\,\nu_R$ carrier.

Anomaly computations are performed only over left-handed Weyl fermions. Therefore physical right-handed fields are converted to left-handed conjugates:

$$f_R \rightsquigarrow f_R^c.$$

The all-left-handed state table is the central object displayed in the front-matter ledger; the appendix verifies every hypercharge entry.

LH field	$SU(3)_c$	$SU(2)_L$	T_R^3	$B-L$	Y	**Multiplicity**
$q_L = (u_L, d_L)$	$\mathbf{3}$	$\mathbf{2}$	0	1/3	1/6	6
$\ell_L = (\nu_L, e_L)$	$\mathbf{1}$	$\mathbf{2}$	0	-1	$-1/2$	2
u_R^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	$-1/2$	$-1/3$	$-2/3$	3
d_R^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	$+1/2$	$-1/3$	$+1/3$	3
ν_R^c	$\mathbf{1}$	$\mathbf{1}$	$-1/2$	$+1$	0	1
e_R^c	$\mathbf{1}$	$\mathbf{1}$	$+1/2$	$+1$	$+1$	1

Proposition 6.1 (Left-handed anomaly ledger resolution)**.** *All anomaly traces in Paper II are traces over*

$$q_L, \quad \ell_L, \quad u_R^c, \quad d_R^c, \quad e_R^c, \quad \nu_R^c,$$

not over an unconverted mixture of physical left- and right-handed fields.

Proof. The all-left-handed convention is theorem A.7. Abelian charge reversal is theorem A.8, and nonabelian conjugation is theorem A.9. The table above is exactly the branch of $W_{\mathrm{fam}}^{\mathrm{LH}}$ under

$$SU(4)_C \times SU(2)_L \times SU(2)_R \longrightarrow SU(3)_c \times SU(2)_L \times U(1)_Y.$$

□

7 Pati–Salam Branching and Hypercharge

The Pati–Salam group is

$$G_{\mathrm{PS}} = SU(4)_C \times SU(2)_L \times SU(2)_R.$$

The $SU(4)_C$ factor contains color and $B-L$:

$$SU(4)_C \supset SU(3)_c \times U(1)_{B-L}.$$

With

$$T_{15} = \frac{1}{2\sqrt{6}}\,\mathrm{diag}(1,1,1,-3),$$

one has

$$B-L = 2\sqrt{\frac{2}{3}}T_{15}, \qquad Q_{BL} = \frac{1}{2}(B-L) = \sqrt{\frac{2}{3}}T_{15}.$$

The color–lepton branchings are

$$\mathbf{4} \to (\mathbf{3})_{1/3} \oplus (\mathbf{1})_{-1}, \qquad \bar{\mathbf{4}} \to (\bar{\mathbf{3}})_{-1/3} \oplus (\mathbf{1})_{+1}.$$

If hypercharge is

$$Y = aT_R^3 + b(B-L),$$

then matching the q_L row gives

$$\frac{1}{6} = b\left(\frac{1}{3}\right), \qquad b = \frac{1}{2}.$$

Matching the u_R^c row then gives

$$-\frac{2}{3} = a\left(-\frac{1}{2}\right) + \frac{1}{2}\left(-\frac{1}{3}\right),$$

so

$$a = 1.$$

Therefore

$$Y = T_R^3 + \frac{1}{2}(B-L). \tag{7.1}$$

Export 7.1 (Pati–Salam charge export)**.** Paper II exports the family submodule

$$W_{\mathrm{fam}}^{\mathrm{LH}} \simeq (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}),$$

its branching to the all-left-handed state table, and the hypercharge map

$$Y = T_R^3 + \frac{1}{2}(B-L).$$

GU III may not alter this table without opening a new anomaly ledger.

8 Minimal-Rank Spin(10) Envelope

The Pati–Salam carrier can also be embedded in a simple chiral envelope. This is not needed to state the low-energy family table, but it is part of the minimal chiral-completion theorem and gives GU III a clean simple-envelope reference for anomaly and representation bookkeeping. The representation-theoretic chain used here is the one-family minimal chiral-completion chain developed in the standalone proof record [20].

Definition 8.1 (Minimal-rank simple chiral envelope)**.** A minimal-rank simple chiral envelope for $W^{\mathrm{LH}}_{\mathrm{fam}}$ is a compact connected simple group G satisfying:

(i) G contains G_{PS} as a subgroup;

(ii) G has an irreducible complex representation R whose restriction to G_{PS} is exactly $W^{\mathrm{LH}}_{\mathrm{fam}}$;

(iii) no spectator matter appears in the restriction;

(iv) the rank of G is minimal among compact connected simple groups satisfying these conditions.

Theorem 8.2 (Minimal-rank Spin(10) envelope)**.** *The minimal-rank compact connected simple chiral envelope of* $W^{\mathrm{LH}}_{\mathrm{fam}}$ *is* Spin(10)*. Its complex chiral spinor representation branches as*

$$\mathbf{16}\downarrow_{SU(4)_C\times SU(2)_L\times SU(2)_R} = (\mathbf{4},\mathbf{2},\mathbf{1})\oplus(\bar{\mathbf{4}},\mathbf{1},\mathbf{2}).$$

Proof. The Pati–Salam group has rank

$$\mathrm{rank}(SU(4)_C)+\mathrm{rank}(SU(2)_L)+\mathrm{rank}(SU(2)_R)=3+1+1=5.$$

Any simple envelope containing it has rank at least five. The rank-five simple compact Lie algebras are A_5, B_5, C_5, and D_5, corresponding to

$$SU(6),\qquad \mathrm{Spin}(11),\qquad \mathrm{Sp}(5),\qquad \mathrm{Spin}(10).$$

The required representation must be complex, chiral, spectator-free on restriction to G_{PS}, and must restrict to

$$(\mathbf{4},\mathbf{2},\mathbf{1})\oplus(\bar{\mathbf{4}},\mathbf{1},\mathbf{2}).$$

The rank-five sweep is:

Rank-five group	**Relevant small representations**	**Obstruction/status**
$SU(6)$	$\mathbf{6},\mathbf{15},\mathbf{20}$	No spectator-free complex **16** restricting to $W^{\mathrm{LH}}_{\mathrm{fam}}$.
Sp(5)	**10** pseudoreal and higher symplectic carriers	Pseudoreal/symplectic carrier, not the complex chiral **16**.
Spin(11)	$\mathbf{11},\mathbf{32}$	Spinor restricts through D_5 as a doubled $\mathbf{16}\oplus\overline{\mathbf{16}}$ sector.
Spin(10)	$\mathbf{16}_+,\mathbf{16}_-$	Chiral spinor restricts exactly to $W^{\mathrm{LH}}_{\mathrm{fam}}$.

For $D_5 = \mathrm{Spin}(10)$, the standard subgroup

$$\mathrm{Spin}(10) \supset \mathrm{Spin}(6) \times \mathrm{Spin}(4) \simeq SU(4)_C \times SU(2)_L \times SU(2)_R$$

gives the chiral spinor branching

$$\mathbf{16} \downarrow_{G_{\mathrm{PS}}} = (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}).$$

This realizes the desired carrier at minimal rank and without spectator states in the restriction. Therefore Spin(10) is the minimal-rank simple chiral envelope in the declared category. □

Corollary 8.3 (Minimal chiral-completion carrier chain). *Inside the declared one-family minimal compact completion category, the carrier chain is*

$$\mathrm{SM} + \nu_R \Longrightarrow \mathrm{span}\{Y, B - L\} \Longrightarrow SU(4)_C \times SU(2)_L \times SU(2)_R \Longrightarrow \mathbf{16}_{\mathrm{Spin}(10)} \Longrightarrow J_5^\mu.$$

The last arrow is the internal-singlet axial-current bridge proved in Section 12.

Proof. The family-universal abelian-direction theorem supplies $\mathrm{span}\{Y, B - L\}$ and selects $B - L$ modulo hypercharge inside the declared category. The minimal color–lepton and right-weak carrier lemmas supply the Pati–Salam block. Theorem 8.2 supplies the minimal-rank simple chiral envelope. Theorem 12.2 identifies the internal-singlet Lorentz axial current consumed by the Paper I axial contact. □

Scope Boundary 8.4 (Envelope status). The Spin(10) envelope theorem is a representation-theoretic completion result. It does not assert gauge unification, a breaking chain, a scalar vacuum, a mass spectrum, or low-energy Spin(10) symmetry. It identifies the minimal simple chiral envelope of the one-family Pati–Salam carrier inside the declared category.

9 Anomaly Ledgers

Paper II proves anomaly closure on $W^{\mathrm{LH}}_{\mathrm{fam}}$. The anomaly ledger follows

$$S_{\mathrm{int}} \longrightarrow W^{\mathrm{LH}}_{\mathrm{fam}} \longrightarrow \text{state table} \longrightarrow \text{anomaly traces.}$$

Anomaly	**Result**	**Proof check**
$SU(3)^3$	0	$2A(\mathbf{3}) + A(\bar{\mathbf{3}}) + A(\bar{\mathbf{3}}) = 0$.
$SU(3)^2U(1)_Y$	0	Explicit sum in theorem F.1.
$SU(2)^2_L U(1)_Y$	0	Explicit sum in theorem F.1.
$U(1)^3_Y$	0	Multiplicity-weighted cubic hypercharge sum.
$\mathrm{grav}^2 U(1)_Y$	0	Multiplicity-weighted hypercharge sum.
Witten $SU(2)_L$	absent	Four left-handed $SU(2)_L$ doublets.
$B - L$ ledger	0	Closes with ν^c_R.
Pati–Salam $SU(4)^3$	0	$2A(\mathbf{4}) + 2A(\bar{\mathbf{4}}) = 0$.
Spin(10) envelope	compatible	Restricts to the spectator-free Pati–Salam family block.
I_6	0	Six-form polynomial cross-check.

Theorem 9.1 (Anomaly closure for the Paper II family)**.** *For the family submodule $W^{\mathrm{LH}}_{\mathrm{fam}}$, all perturbative Standard Model anomalies vanish:*

$$\mathcal{A}_{SU(3)^3} = \mathcal{A}_{SU(3)^2U(1)_Y} = \mathcal{A}_{SU(2)^2_L U(1)_Y} = \mathcal{A}_{U(1)^3_Y} = \mathcal{A}_{\mathrm{grav}^2 U(1)_Y} = 0.$$

The global $SU(2)_L$ Witten anomaly is absent, the full $B - L$ anomaly ledger closes with ν^c_R, the Pati–Salam block is locally and globally anomaly-safe, the Spin(10) *envelope restricts to the same spectator-free one-family block, and*

$$I_6^{\text{one family}} = 0.$$

Proof. The perturbative Standard Model sums are theorem F.1. The Witten check is theorem F.2. The $B - L$ ledger is theorem F.3. The Pati–Salam anomaly check is theorem F.5. The Spin(10) envelope compatibility is theorem F.6. The six-form theorem is theorem F.7. □

Corollary 9.2 (Anomaly-safe input for the quantum legality layer)**.** *The one-family matter block exported by Paper II is an anomaly-safe input for GU III. GU III may add BRST/BV, boundary, counterterm, RG/matching, and sign-corridor legality checks, but it may not change the all-left-handed state table or the hypercharge ledger without reopening the anomaly proof.*

Proof. The anomaly closure theorem proves the perturbative, global, $B - L$, Pati–Salam, Spin(10)-restriction, and six-form anomaly checks for the exported one-family block. Since these checks are representation-dependent, changing the state table or hypercharge ledger changes the anomaly problem. Thus the representation table is read-only for GU III unless a new anomaly ledger is supplied. □

10 Gauge-Admissible Scalar Channels and Yukawa Seeds

Paper II proves the gauge-theoretic selection rule for Yukawa seeds and identifies the allowed Pati–Salam scalar channels. It does not assert the existence of a specific geometric scalar mode unless the relevant internal geometric module is separately declared.

Theorem 10.1 (Yukawa seed selection)**.** *A Yukawa seed $\psi_a \psi_b \phi$ is gauge-allowed if and only if*

$$R_a \otimes R_b \otimes R_\phi \supset \mathbf{1}.$$

For the Pati–Salam family submodule,

$$\overline{\Psi}_L \Psi_R \sim (\mathbf{1}, \mathbf{2}, \mathbf{2}) \oplus (\mathbf{15}, \mathbf{2}, \mathbf{2}).$$

Thus the minimal scalar channel is

$$\Phi_H \sim (\mathbf{1}, \mathbf{2}, \mathbf{2}),$$

with optional nonminimal channel $(\mathbf{15}, \mathbf{2}, \mathbf{2})$.

Proof. The general selection rule is theorem G.1. The Pati–Salam bilinear decomposition is theorem G.2. The bi-doublet singlet contraction is theorem G.3. □

Corollary 10.2 (Spin(10) envelope does not alter the GU II minimal Yukawa ledger)**.** *Passing from the Pati–Salam family block to its minimal-rank* Spin(10) *envelope does not alter the GU II minimal slice Yukawa seed ledger. The GU II minimal seed channel remains*

$$(\mathbf{1}, \mathbf{2}, \mathbf{2}),$$

with $(\mathbf{15}, \mathbf{2}, \mathbf{2})$ *retained as the nonminimal Pati–Salam channel.*

Proof. The Spin(10) envelope restricts to the same spectator-free G_{PS} family block. The GU II Yukawa ledger is a representation statement on that restricted block. Additional Spin(10)-level scalar representations require a separate scalar-module declaration and are not part of the minimal GU II seed export. □

Scope Note 10.3 (Scalar-channel boundary)**.** Paper II proves selection rules, singlet existence, and candidate scalar seed channels. It does not export a numerical fermion mass texture, scalar potential, or proof that a specific internal geometric fluctuation contains $(\mathbf{1}, \mathbf{2}, \mathbf{2})$. Such a claim requires a declared internal geometric module and a decomposition theorem.

11 Gauge Sector on the Slice

Paper II proves the gauge-normalization ledger used by GU III and GU IV. For simple factors,

$$\mathrm{tr}_{\mathbf{fund}}(T_a T_b) = \frac{1}{2}\delta_{ab}.$$

For the $SU(4)_C$ generator T_{15},

$$Q_{BL} = \frac{1}{2}(B - L) = \sqrt{\frac{2}{3}}T_{15},$$

so

$$g_{BL} = \sqrt{\frac{3}{2}}g_4.$$

The neutral covariant derivative along $SU(2)_R \times U(1)_{B-L}$ is

$$D_\mu = \partial_\mu + ig_R T_R^3 A_\mu^R + ig_{BL} Q_{BL} A_\mu^{BL}.$$

If a breaking vacuum preserves $Y = T_R^3 + Q_{BL}$, then the massless hypercharge gauge field is

$$B_\mu^Y = \frac{g_Y}{g_R}A_\mu^R + \frac{g_Y}{g_{BL}}A_\mu^{BL},$$

with

$$\frac{1}{g_Y^2} = \frac{1}{g_R^2} + \frac{1}{g_{BL}^2}.$$

Theorem 11.1 (Pati–Salam unification normalization)**.** *If $g_R = g_4 = g$ at a Pati–Salam unification scale, then*

$$g_Y^2 = \frac{3}{5}g^2, \qquad g_1^2 = \frac{5}{3}g_Y^2 = g^2.$$

Proof. The normalization $g_{BL} = \sqrt{3/2}\, g_4$ gives $g_{BL}^2 = 3g^2/2$. Therefore

$$\frac{1}{g_Y^2} = \frac{1}{g_R^2} + \frac{1}{g_{BL}^2} = \frac{1}{g^2} + \frac{2}{3g^2} = \frac{5}{3g^2}.$$

Thus $g_Y^2 = 3g^2/5$, and the $SU(5)$-normalized coupling $g_1^2 = (5/3)g_Y^2$ equals g^2. □

Export 11.2 (Gauge normalization export)**.** Paper II exports

$$Y = T_R^3 + \frac{1}{2}(B - L), \qquad Q_{BL} = \sqrt{\frac{2}{3}}T_{15}, \qquad g_{BL} = \sqrt{\frac{3}{2}}g_4,$$

and

$$\frac{1}{g_Y^2} = \frac{1}{g_R^2} + \frac{1}{g_{BL}^2}.$$

Scope Boundary 11.3 (Gauge-normalization boundary)*.* The gauge-normalization export fixes the generator and coupling conventions used by GU III and GU IV. It does not assert a specific symmetry-breaking scalar potential, heavy gauge-boson mass spectrum, or low-energy unification prediction.

12 Internal-Singlet Axial-Current Bridge

The representation-theoretic carrier determines which internal current can couple to minimal Einstein–Cartan torsion. Paper I supplies the local axial/torsion law. Paper II supplies the matter-current identification of the Lorentz axial current that feeds that law.

Definition 12.1 (Internal-singlet Lorentz axial current)**.** On the one-family carrier, the internal-singlet Lorentz axial current is

$$J_5^\mu = \sum_{I=1}^{16} \bar{\psi}_I \gamma^\mu \gamma^5 \psi_I, \tag{12.1}$$

where the four-component notation denotes the standard Lorentz axial bilinear associated to the chiral carrier and does not add spectator internal states.

Theorem 12.2 (Minimal ECSK torsion selects the internal-singlet axial current)**.** *Minimal ECSK auxiliary torsion couples to the internal-singlet Lorentz axial current* (12.1)*. It does not couple minimally to hypercharge, $B-L$, T_R^3, or any other internal gauge current.*

Proof. The chimeric carrier separates Lorentz and internal factors:

$$S_Y|_X \simeq S_X \otimes_{\mathrm{gr}} S_{\mathrm{int}}.$$

Lorentz gamma matrices act on S_X, while internal generators act on the internal carrier. Thus

$$\gamma^\mu\gamma^5 = \gamma^\mu\gamma^5 \otimes \mathbf{1}_{\mathrm{int}}, \qquad T_A = \mathbf{1}_{S_X} \otimes T_A.$$

Minimal ECSK torsion is a spacetime-geometric auxiliary field. It carries no Pati–Salam, hypercharge, $B-L$, or T_R^3 index. Therefore its minimal spin-current source is the internal singlet in the Lorentz axial bilinear. In an orthonormal internal basis $\{e_I\}_{I=1}^{16}$, write

$$\Psi = \sum_{I=1}^{16} \psi_I \otimes e_I.$$

The internal-singlet projection is the trace over the internal indices:

$$\left(\bar{\Psi}\gamma^\mu\gamma^5\Psi\right)_{\mathbf{1}} = \sum_{I=1}^{16} \bar{\psi}_I \gamma^\mu\gamma^5\psi_I.$$

This is (12.1). Internal-adjoint or higher current components would require internal-indexed torsion-like fields or nonminimal couplings, which are separate extension channels. Hence the minimal torsion channel selects the internal-singlet Lorentz axial current. □

Corollary 12.3 (Matter-current input to the Paper I axial contact)**.** *On the one-family matter carrier, the Paper I axial/torsion law*

$$\Delta L_X = C_{55} O_{55}, \qquad O_{55} = -J_{5\mu} J_5^\mu,$$

uses the internal-singlet Lorentz axial current (12.1).

Proof. Paper I proves the local axial/torsion law for an admissible Lorentz axial current. Theorem 12.2 identifies the current supplied by the one-family internal carrier. Substituting that current into the Paper I operator basis gives the displayed contact. □

Scope Boundary 12.4 (Current identification versus current conservation)*.* The axial-current bridge identifies the current to which minimal torsion couples. It does not prove that the microscopic axial current is conserved. Conservation, effective conservation, anomaly suppression, mass/Yukawa suppression, source histories, sterile dense-fermion branch status, and washout survival are branch-domain data handled in the current ledger and certified for downstream use by GU III.

13 Slice Currents, Axial Number, and Thermal Ledger

Define the vector and axial currents by

$$J^\mu = \bar{\Psi}\gamma^\mu\Psi, \qquad J_5^\mu = \bar{\Psi}\gamma^\mu\gamma^5\Psi.$$

For a future-directed unit timelike observer u^μ, define

$$n_5 = -u_\mu J_5^\mu.$$

In a homogeneous comoving frame,

$$J_5^\mu = n_5 u^\mu, \qquad J_{5\mu}J_5^\mu = -n_5^2, \qquad O_{55} = n_5^2.$$

Paper II distinguishes conserved, anomalous, and source-driven axial-current classes:

$$\nabla_\mu J_5^\mu = 0, \tag{13.1}$$

$$\nabla_\mu J_5^\mu = \mathcal{S}_{\text{anom}}, \tag{13.2}$$

$$\nabla_\mu J_5^\mu = \mathcal{S}_{\text{source}}. \tag{13.3}$$

Domain Condition 13.1 (Conserved axial branch admissibility)**.** The conserved-branch export

$$n_5(a) = n_{5,0}a^{-3}$$

is available only for an axial combination whose anomaly coefficient, explicit mass/Yukawa violation, and source terms vanish or are dynamically negligible on the interval of use. Symbolically,

$$\nabla_\mu J_5^\mu = \sum_i c_i \frac{g_i^2}{16\pi^2} F_i\widetilde{F}_i + 2im\bar{\psi}\gamma^5\psi + \mathcal{S}_{\text{int}}. \tag{13.4}$$

The conserved branch is the domain where the right-hand side of (13.4) is zero or negligible for the purpose of the effective evolution.

If the current is conserved in FLRW, then $n_5(a) = n_{5,0}a^{-3}$, as proved in theorem I.3. If it is sourced, then

$$\frac{d}{dt}(a^3 n_5) = a^3\mathcal{S}(t),$$

as proved in theorem I.4.

The thermal reference formulas are

$$\rho(T) = \frac{\pi^2}{30}g_*(T)T^4, \qquad n_f(T) = \frac{3\zeta(3)}{4\pi^2}g_f T^3,$$

and for small chemical potential,

$$\Delta n_f = \frac{g_f}{6}\mu T^2 + O(\mu^3).$$

An axial thermal population requires declared axial weights w_i:

$$g_{5,\text{eff}} = \sum_i w_i g_i, \qquad n_{5,\text{ref}}(T) = \frac{3\zeta(3)}{4\pi^2} g_{5,\text{eff}} T^3.$$

With $C_{55} > 0$, the conserved-current stress tensor gives

$$\rho_5 = p_5 = -C_{55} n_5^2.$$

For a conserved branch,

$$\sigma_0^2 = C_{55} n_{5,0}^2, \qquad \rho_5(a) = p_5(a) = -\sigma_0^2 a^{-6}, \qquad w_5 = 1.$$

This conserved branch is NEC-violating:

$$\rho_5 + p_5 = -2\sigma_0^2 a^{-6} < 0.$$

Its admissibility as a bounce-supporting or observable cosmological branch is a downstream GU III/GU IV/GU V domain question.

13.1 Sterile dense-fermion source branch

The conserved homogeneous branch is not the only current lane exported by Paper II. Paper II also defines a source-class branch for species-resolved weak-singlet dense fermions. This branch is called the sterile dense-fermion branch. It is a matter/current source class, not a GU III legality certificate and not a detector-amplitude theorem.

Definition 13.2 (Sterile dense-fermion branch)**.** A sterile dense-fermion branch is a species-resolved occupied Dirac sector satisfying, on the interval of use,

$$T_{\text{spin}} \ll \bar{\mu}, \qquad \frac{\mu_5}{\bar{\mu}} \gtrsim 0.27, \qquad \Gamma_\chi \tau_{\text{prod}} \lesssim 1, \qquad \rho_{\text{deg}} \lesssim \rho_b.$$

The carrier is weak-singlet/sterile or otherwise sphaleron-blind over the production interval.

Theorem 13.3 (Degenerate branch axial-polarization export)**.** *For a degenerate branch with*

$$n_\pm = \frac{g_p}{6\pi^2} p_{F\pm}^3, \qquad P_A = \frac{n_+ - n_-}{n_+ + n_-},$$

one has

$$P_A = \frac{p_{F+}^3 - p_{F-}^3}{p_{F+}^3 + p_{F-}^3}.$$

Writing $r = p_{F+}/p_{F-}$, this becomes

$$P_A(r) = \frac{r^3 - 1}{r^3 + 1}.$$

The threshold $P_A \gtrsim 0.68$ is reached for $r \gtrsim 1.74$.

Proof. This is theorem I.11. Substituting the degenerate number densities into $P_A = (n_+ - n_-)/(n_+ + n_-)$ cancels the common factor $g_p/(6\pi^2)$. Setting $p_{F+} = rp_{F-}$ gives $P_A(r) = (r^3 - 1)/(r^3 + 1)$. Solving $P_A(r) \geq 0.68$ gives $r^3 \geq 5.25$, hence $r \gtrsim 1.74$. □

Domain Condition 13.4 (Sterile dense-fermion branch admissibility)**.** Downstream use of the sterile dense-fermion branch requires a declared occupied species list, weak-singlet/sterile carrier status, degeneracy g_p, chemical-potential split, chirality-flip ledger, production interval, and energy-budget ledger. The representative energy gate is

$$\frac{\rho_{\rm deg}}{\rho_b} \simeq 0.111 \frac{g_p}{g_*} \left(\frac{\bar{\mu}}{T_b}\right)^4$$

at $\mu_5/\bar{\mu} \simeq 0.27$. This energy gate is an admissibility condition, not an observational preference. A source packet whose degenerate-sector energy exceeds the bounce budget is outside the sterile dense-fermion branch, even if its axial polarization and formal current algebra are otherwise well-defined. Averaged or saturation-level spin-fluid estimates may be used as control branches only; they do not define the source-resolved sterile dense-fermion packet unless the occupied species list, washout gate, and energy gate are all satisfied.

The washout gate is

$$\Gamma_\chi \tau_{\rm prod} \lesssim 1.$$

For mass-suppressed chirality or spin flips,

$$\Gamma_\chi \sim \left(\frac{m_{\rm eff}}{\bar{\mu}}\right)^2 \Gamma_{\rm scatt}.$$

Scope Boundary 13.5 (Sterile dense-fermion branch status)*.* GU II defines the sterile dense-fermion branch as a matter/current source class. GU III certifies anomaly, washout, BRST/BV, boundary, and sign-corridor legality. GU IV maps declared source-adapter packets to observables only after the branch gates have been declared. GU V audits the declared packet, its domain, and its observable-interface status. GU II does not compute a universal parent-collapse value, tensor amplitude, detector corridor, or observational fit.

Export 13.6 (Sterile dense-fermion branch export)**.** Paper II exports

$$\text{GU2-K} = \left\{ \begin{array}{c} \text{occupied species list}, \quad \text{weak-singlet/sterile carrier status}, \quad T_{\rm spin}, \quad \bar{\mu}, \\ \mu_5, \quad P_A(r), \quad \Gamma_\chi \tau_{\rm prod}, \quad \rho_{\rm deg}/\rho_b \end{array} \right\}.$$

This is source-class data. It is not a universal value of σ_0, not a detector amplitude, and not a legality certificate without GU III. A packet failing the energy gate $\rho_{\mathrm{deg}} \lesssim \rho_b$ is not an admissible instance of GU2-K.

Export 13.7 (Axial current export)**.** Paper II exports the current definitions, axial density convention, spin-density convention, thermal degeneracy factors, axial population weights, conservation/source classes, conserved-branch admissibility condition, conserved-branch σ_0 handoff, and the sterile dense-fermion branch source-class packet GU2-K. GU IV may use the conserved branch only when conservation or effective conservation is declared; downstream source-adapter use of GU2-K requires GU III legality status, a species-resolution packet, and satisfaction of the washout and energy-admissibility gates.

14 Handoff to GU III and GU IV

The immediate downstream consumers are GU III and GU IV. GU III consumes the spin-domain carrier, family submodule, projector, state table, anomaly ledger, Yukawa selection, gauge normalization, current basis, Spin(10) envelope reference, and sterile dense-fermion branch legality gates. GU IV consumes the current and thermal ledger, including n_5, $g_{5,\mathrm{eff}}$, conservation classes, and the conserved-branch σ_0 handoff. GU V sees these results through the GU IV observable packet.

Table 2: GU II downstream handoff.

Downstream paper	**Input consumed from GU II**	**Status**
GU III	S_{int}, W_{fam}, Π_{fam}, state table, anomaly ledger, Yukawa selection, gauge normalization, current basis, Spin(10) envelope reference, and sterile dense-fermion branch legality gates	mandatory
GU IV	J_5^μ, n_5, $g_{5,\mathrm{eff}}$, thermal scalings, conservation/source assumptions, $\sigma_0^2 = C_{55} n_{5,0}^2$ on conserved branches, and declared source-adapter data for GU2-K only after GU III legality	mandatory / conditional
KT0/KTC	sterile dense-fermion branch source-class data, if declared, as input to species-resolution and descendant-amplitude calculations	conditional
GU V	imports the GU IV packet; may cite GU II for source provenance	indirect

The formal downstream import protocol is theorem J.5. Reproducibility requires artifact manifest entries with filename, version, date, checksum, and proof label, as stated in theorem J.4.

15 Domain Boundaries and Review Criteria

This section records the exact domain boundaries of the Paper II matter packet. Each boundary corresponds to a theorem hypothesis, ledger entry, or artifact condition.

F1. Spin-domain requirement. If no global spin, Spin^c, or local tubular factorization domain is available, then S_{int} is not defined in the sense required by Paper II.

F2. Minimal chiral-completion scope. The Pati–Salam and Spin(10) conclusions hold inside the declared one-family minimal compact completion category. Nonminimal, spectator-containing, multi-family, or alternative internal geometries require a separate completion ledger.

F3. Projector compatibility requirement. If $[\Pi_{\text{fam}}, \rho_{\text{PS}}(g)] \neq 0$ for some g, then selecting the family submodule breaks or restricts the gauge action.

F4. State-table stability. If the state table changes, the anomaly ledger must be reopened.

F5. Hypercharge compatibility. The charge table must be compatible with $Y = T_R^3 + (B - L)/2$.

F6. Anomaly closure. Any uncancelled perturbative, global, $B - L$, Pati–Salam, Spin(10)-restriction, or six-form anomaly-polynomial obstruction invalidates the Paper II matter packet.

F7. Yukawa scope. Paper II proves selection rules and scalar seed channels. Numerical textures require additional internal spectral data.

F8. Gauge-normalization stability. Changing Q_{BL}, g_{BL}, or the hypercharge kernel relation requires a new normalization ledger.

F9. Current-branch admissibility. The conserved-branch result $n_5(a) = n_{5,0}a^{-3}$ may be used only for a conserved or effectively conserved axial branch.

F10. Sterile dense-fermion branch admissibility. The sterile dense-fermion branch may be exported only with a declared occupied species list, weak-singlet/sterile carrier status, degeneracy, chemical-potential split, polarization, washout ledger, production interval, and energy-budget ledger. GU III legality is required before downstream use as an active source packet.

F11. Artifact manifest requirement. A table, worksheet, YAML ledger, or notebook may be called released only if it has a manifest entry with filename, version, date, checksum, and proof label.

Review Criterion 15.1 (Main-text proof support)**.** Every claim in the main text must point to an appendix theorem, proposition, lemma, definition, export, domain condition, scope boundary, or review criterion. If no such proof object exists, the claim is not part of the Paper II proof spine. In particular, claims about the sterile dense-fermion branch must point to GU2-K, the branch admissibility gates, and the GU III legality handoff.

A Conventions, Classical Inputs, and Matter-Ledger Notation

This appendix fixes the convention identities used by the Paper II matter ledger. It also fixes the local notation used when the one-family minimal chiral-completion theorem is imported into the GU II matter/current ledger. The completion chain used below is developed in the standalone proof record [20]; this paper incorporates that chain as part of the GU II export ledger.

A.1 Completed variables consumed by the matter ledger

The completed gauge connection is

$$\widehat{A} = A - B. \tag{A.1}$$

The curvature convention is

$$F(A) = dA + A \wedge A, \tag{A.2}$$

and the covariant derivative of the adjoint-valued one-form B is

$$D_A B = dB + A \wedge B + B \wedge A. \tag{A.3}$$

Proposition A.1 (Completed curvature identity)**.** *With these conventions,*

$$\widehat{F} := F(\widehat{A}) = F(A) - D_A B + B \wedge B. \tag{A.4}$$

Proof. Expanding $F(A - B)$ gives

$$\begin{aligned} F(A - B) &= d(A - B) + (A - B) \wedge (A - B) \\ &= dA - dB + A \wedge A - A \wedge B - B \wedge A + B \wedge B \\ &= F(A) - (dB + A \wedge B + B \wedge A) + B \wedge B \\ &= F(A) - D_A B + B \wedge B. \end{aligned} \tag{A.5}$$

□

Convention Note A.2 (Completed-curvature sign). With $\widehat{A} = A - B$ and $D_A B = dB + A \wedge B + B \wedge A$, the quadratic term in $\widehat{F}$ is $+B \wedge B$. A formula with $-B \wedge B$ uses a different convention.

The completed spin connection and augmented torsion consumed by the matter ledger are

$$\widehat{\omega}_B = \omega - \Upsilon_e(B), \qquad T_{\mathrm{aug}} = T(\widehat{\omega}_B).$$

The completed Levi–Civita slice statement is

$$\mathrm{pr}_{TX}\, \iota^* T_{\mathrm{aug}} = 0 \quad \Rightarrow \quad \widehat{\omega}_{X,B} = \omega_{\mathrm{LC}}(g_X).$$

A.2 Metric, gamma matrices, chirality, and currents

The physical observation slice X has signature $(-,+,+,+)$. Slice gamma matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g_X^{\mu\nu}\mathbf{1}.$$

The chirality matrix satisfies

$$(\gamma^5)^2 = 1, \qquad \{\gamma^5, \gamma^\mu\} = 0.$$

The chiral projectors are

$$P_L = \frac{1-\gamma^5}{2}, \qquad P_R = \frac{1+\gamma^5}{2}.$$

Definition A.3 (Vector and axial currents)**.** For a Dirac spinor Ψ on the slice, define

$$J^\mu = \bar{\Psi}\gamma^\mu\Psi, \qquad J_5^\mu = \bar{\Psi}\gamma^\mu\gamma^5\Psi.$$

For a future-directed unit observer u^μ, define

$$n_5 = -u_\mu J_5^\mu, \qquad O_{55} = -J_{5\mu}J_5^\mu.$$

Lemma A.4 (Timelike axial-current sign)**.** *If $J_5^\mu = n_5 u^\mu$ and $u_\mu u^\mu = -1$, then*

$$J_{5\mu}J_5^\mu = -n_5^2, \qquad O_{55} = n_5^2.$$

Proof. Using $J_5^\mu = n_5 u^\mu$,

$$J_{5\mu}J_5^\mu = n_5^2 u_\mu u^\mu = -n_5^2.$$

The result follows from $O_{55} = -J_{5\mu}J_5^\mu$. □

When a totally antisymmetric spin density is used, Paper II adopts

$$s^{\mu\nu\lambda} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}J_{5\rho}.$$

Definition A.5 (Internal-singlet Lorentz axial current)**.** Let W_{fam} be the one-family internal carrier. The internal-singlet Lorentz axial current is the sum of the Lorentz axial bilinear over the internal family carrier,

$$J_5^\mu = \sum_{\alpha \in W_{\text{fam}}} \bar{\Psi}_\alpha \gamma^\mu \gamma^5 \Psi_\alpha,$$

or, equivalently, the trace over the identity operator on the internal family space. It is a Lorentz axial current and an internal gauge singlet.

Proposition A.6 (Minimal ECSK torsion selects the internal-singlet axial current)**.** *In the minimally coupled Einstein–Cartan–Sciama–Kibble spinor sector, algebraic axial torsion couples to the Lorentz axial current in theorem A.5. It does not couple to hypercharge, $B-L$, T_R^3, or any other internal gauge current as the minimal torsion source.*

Proof. The algebraic Cartan equation couples torsion to the spin current obtained by varying the spin connection. For minimally coupled spinors, the totally antisymmetric torsion channel is sourced by the Lorentz axial bilinear $\bar{\Psi}\gamma^\mu\gamma^5\Psi$. The spin connection acts on Lorentz/Clifford indices, while hypercharge, $B-L$, and T^3_R act on internal gauge labels. Therefore the minimal torsion source is the internal trace of the Lorentz axial bilinear, i.e. the internal-singlet axial current. An internal generator inserted into the current would define a gauge-current-weighted axial object; it is not the minimal ECSK torsion source. □

A.3 All-left-handed anomaly convention

Definition A.7 (All-left-handed anomaly convention)**.** The Paper II anomaly ledger uses

$$f_R \rightsquigarrow f_R^c \in \text{left-handed conjugate representation.}$$

For example,

$$u_R : (\mathbf{3},\mathbf{1})_{2/3} \rightsquigarrow u_R^c : (\bar{\mathbf{3}},\mathbf{1})_{-2/3}.$$

Lemma A.8 (Charge conjugation reverses abelian charges)**.** *If a physical right-handed field f_R has abelian charge q, then f_R^c has charge $-q$.*

Proof. Under a $U(1)$ transformation, $f_R \mapsto e^{iq\alpha} f_R$. Its conjugate transforms as $f_R^c \mapsto e^{-iq\alpha} f_R^c$. □

Lemma A.9 (Conjugation reverses complex nonabelian representations)**.** *If a physical right-handed field transforms in a complex nonabelian representation R, then its left-handed conjugate transforms in $\bar{R}$. For pseudoreal representations, such as the $SU(2)$ doublet, the conjugate representation is equivalent to the original doublet.*

Proof. A field in representation R transforms as $f \mapsto R(g)f$. Its conjugate transforms by the complex conjugate representation $\bar{R}$. Pseudoreal representations are identified with their conjugates by an invariant antisymmetric tensor. □

A.4 Trace and anomaly-coefficient conventions

For a simple nonabelian factor G,

$$\mathrm{tr}_{\mathbf{fund}}(T_a T_b) = \frac{1}{2}\delta_{ab}.$$

The Dynkin index $T(R)$ is defined by

$$\mathrm{tr}_R(T_a T_b) = T(R)\delta_{ab},$$

so $T(\mathbf{N}) = T(\bar{\mathbf{N}}) = 1/2$ for $SU(N)$.

Lemma A.10 (Dynkin index of conjugate representations)**.** *For a unitary representation R, $T(\bar{R}) = T(R)$.*

Proof. The conjugate representation generators may be written $T_a^{\bar{R}} = -(T_a^R)^*$. The trace $\mathrm{tr}_{\bar{R}}(T_a^{\bar{R}} T_b^{\bar{R}})$ equals the complex conjugate of $\mathrm{tr}_R(T_a^R T_b^R)$, which is real in the chosen normalization. □

For cubic anomalies,

$$\mathrm{tr}_R(T_a\{T_b, T_c\}) = A(R) d_{abc}.$$

Lemma A.11 (Cubic anomaly coefficient of conjugate representations)**.** *For a complex representation R, $A(\bar{R}) = -A(R)$.*

Proof. Using $T_a^{\bar{R}} = -(T_a^R)^*$, the symmetric cubic trace changes sign under complex conjugation. The invariant coefficient is real, so $A(\bar{R}) = -A(R)$. □

Lemma A.12 (No perturbative cubic anomaly for $SU(2)$)**.** *The perturbative $SU(2)^3$ anomaly vanishes for $SU(2)$ representations.*

Proof. The local cubic gauge anomaly is proportional to $d_{abc} = \mathrm{tr}(T_a\{T_b, T_c\})$. For $SU(2)$, $\{T_b, T_c\} \propto \delta_{bc}\mathbf{1}$, and $\mathrm{tr}\, T_a = 0$. □

Export A.13 (Convention export)**.** Paper II exports the mostly-plus slice metric convention, the gamma-matrix and chirality convention, the all-left-handed anomaly convention, the trace normalization, the conjugation rules $T(\bar{R}) = T(R)$, $A(\bar{R}) = -A(R)$, and the axial-current basis $n_5 = -u_\mu J_5^\mu$, $O_{55} = n_5^2$ for homogeneous timelike currents. It also exports the internal-singlet Lorentz axial-current convention: minimal ECSK torsion couples to the Lorentz axial singlet over the internal carrier, not to hypercharge, $B - L$, or T_R^3 currents.

B Spin and Spin^c Factorization

Let $\iota : X \hookrightarrow Y$ be the observation embedding. Along X, assume an orthogonal splitting of oriented metric bundles

$$TY|_X \simeq TX \oplus \nu.$$

The Whitney product formula gives

$$w_2(TY)|_X = w_2(TX) + w_2(\nu).$$

Lemma B.1 (Spin compatibility along the slice)**.** *Assume $TY|_X \simeq TX \oplus \nu$ and all bundles are oriented. If two of $TY|_X$, TX, and ν are spin, then the third is spin.*

Proof. An oriented real vector bundle is spin iff its second Stiefel–Whitney class vanishes. The Whitney relation forces the third class to vanish whenever the other two vanish. □

Theorem B.2 (Global spin factorization)**.** *Assume $TY|_X \simeq TX \oplus \nu$ as oriented metric bundles, TX and ν carry spin structures, and the spin structure on $TY|_X$ is the Whitney-product spin structure. Then*

$$S_Y|_X \simeq S_X \otimes_{\mathrm{gr}} S_\nu.$$

With $S_{\mathrm{int}} = S_\nu$, this is $S_Y|_X \simeq S_X \otimes_{\mathrm{gr}} S_{\mathrm{int}}$.

Proof. The orthogonal splitting gives $\mathrm{Cl}_{\mathbb{C}}(TX \oplus \nu) \simeq \mathrm{Cl}_{\mathbb{C}}(TX)\widehat{\otimes}\mathrm{Cl}_{\mathbb{C}}(\nu)$. The Whitney-product spin structure sends the product of the two spin principal bundles to the spin bundle of the direct sum. Associated complex Dirac modules therefore factor as the graded tensor product. □

Theorem B.3 (Spin^c factorization)**.** *Assume TX and ν carry Spin^c structures with determinant lines L_X and L_ν. Then $TY|_X$ carries the product Spin^c structure with determinant line*

$$L_Y|_X \simeq L_X \otimes L_\nu,$$

and

$$S^c_Y|_X \simeq S^c_X \otimes_{\mathrm{gr}} S^c_\nu.$$

Proof. The determinant line of the product structure has $c_1(L_X \otimes L_\nu) = c_1(L_X) + c_1(L_\nu) \equiv w_2(TX) + w_2(\nu) = w_2(TY)|_X \pmod 2$. The complex Clifford factorization is the same as in the spin case, with determinant-line twisting carried by the product line. □

Theorem B.4 (Local chimeric factorization)**.** *Let U be a contractible tubular patch meeting X, with restricted orthogonal splitting. Then*

$$S_Y|_{U\cap X} \simeq S_X|_{U\cap X} \otimes_{\mathrm{gr}} S_{\mathrm{int}}|_{U\cap X}.$$

Proof. On a contractible patch, the relevant real vector bundles are trivializable and local spin frames exist. The local Clifford algebra factorization gives the local spinor module factorization. □

Theorem B.5 (Chimeric factorization domain theorem)**.** *Every chimeric factorization used by Paper II belongs to exactly one of the following domains: global spin,* Spin^c*, or local tubular EFT. No chimeric factorization claim is valid without one of these domains being declared.*

Proof. The three cases are precisely theorems B.2 to B.4. These exhaust the spin-domain assumptions used by Paper II. □

Export B.6 (Spin-domain export)**.** Paper II exports the global spin, Spin^c, and local tubular chimeric carrier domains. These supply S_{int}. The internal carrier is then constrained by the one-family minimal chiral-completion category in Sections D and E.

C Gamma-Matrix Split and Projectors

Assume $TY|_X \simeq TX \oplus \nu$. Let

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbf{1}, \qquad \{\kappa_i, \kappa_j\} = 2g^{\mathrm{int}}_{ij}\mathbf{1}, \qquad (\gamma^5)^2 = 1, \qquad \{\gamma^5, \gamma_\mu\} = 0.$$

Define

$$\Gamma_\mu = \gamma_\mu \otimes \mathbf{1}, \qquad \Gamma_i = \gamma^5 \otimes \kappa_i.$$

Theorem C.1 (Gamma-matrix split)**.** *The matrices Γ_μ and Γ_i satisfy the Clifford algebra of $TX \oplus \nu$:*

$$\{\Gamma_A, \Gamma_B\} = 2g_{AB}\mathbf{1}.$$

Proof. Tangential and normal components follow directly from the two Clifford algebras and $(\gamma^5)^2 = 1$. Mixed components vanish because $\{\gamma_\mu, \gamma^5\} = 0$ and the direct sum is orthogonal. □

Corollary C.2 (Internal Clifford action preserves slice chirality)**.** *The internal Clifford generators commute with $\gamma^5 \otimes \mathbf{1}$, so they preserve the P_L/P_R decomposition.*

Proof. $[\gamma^5 \otimes \mathbf{1}, \gamma^5 \otimes \kappa_i] = 0$. □

Lemma C.3 (Chirality projectors are complementary idempotents)**.** *The operators $P_L = (1 - \gamma^5)/2$ and $P_R = (1 + \gamma^5)/2$ satisfy*

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0, \quad P_L + P_R = 1.$$

Proof. This follows immediately from $(\gamma^5)^2 = 1$. □

Lemma C.4 (Slice chirality commutes with internal gauge action)**.** *For an internal action $\mathbf{1}_{S_X} \otimes \rho_{\mathrm{int}}(g)$, both P_L and P_R commute with the internal gauge action.*

Proof. For example, $(P_L \otimes 1)(1 \otimes \rho) = P_L \otimes \rho = (1 \otimes \rho)(P_L \otimes 1)$. □

Proposition C.5 (Slice chirality is independent of internal representation labels)**.** *The slice Weyl decomposition commutes with the internal gauge representation on S_{int}.*

Proof. This is theorem C.4 applied to the image decomposition of P_L and P_R. □

Theorem C.6 (Internal projector compatibility)**.** *Let Π_{int} be an orthogonal projector on a Hermitian internal representation space, and let $W = \Pi_{\mathrm{int}} S_{\mathrm{int}}$. If the unbroken gauge group acts unitarily, then*

$$W \textit{ is gauge invariant} \iff [\Pi_{\mathrm{int}}, \rho_{\mathrm{int}}(g)] = 0 \quad \textit{for all } g.$$

Proof. If the projector commutes with the action, the image is invariant. Conversely, if the image is invariant and the action is unitary, the orthogonal complement is invariant too; the representation is block diagonal with respect to $W \oplus W^\perp$, so the orthogonal projector commutes with the action. □

Theorem C.7 (Gauge compatibility of the family projector)**.** *The family projector satisfies*

$$[\Pi_{\mathrm{fam}}, \rho_{\mathrm{PS}}(g)] = 0 \qquad \textit{for all } g \in G_{\mathrm{PS}}.$$

Proof. By theorem D.11, W_{fam} is a G_{PS}-submodule of S_{int}. Apply theorem C.6 with $\Pi_{\mathrm{int}} = \Pi_{\mathrm{fam}}$. □

Export C.8 (Gamma/projector export)**.** Paper II exports the gamma split, slice chiral projectors, internal-projector compatibility criterion, and family-projector consequence $[\Pi_{\mathrm{fam}}, \rho_{\mathrm{PS}}(g)] = 0$.

D One-Family State Table and Minimal Chiral Completion

This appendix upgrades the family block from a purely declared Pati–Salam module to the selected carrier inside a declared one-family minimal chiral-completion category. The selection ladder follows the standalone proof record [20]. The scope is categorical: the conclusion holds inside the one-family, family-universal, spectator-free, minimal compact completion problem stated below. It is not a claim of uniqueness among all possible internal geometries.

Definition D.1 (One-family chiral input)**.** The one-family chiral input is the Standard Model one-family spectrum with a right-handed neutrino, written in all-left-handed notation:

$$q_L, \quad \ell_L, \quad u_R^c, \quad d_R^c, \quad e_R^c, \quad \nu_R^c.$$

The total number of left-handed Weyl states is 16.

Definition D.2 (One-family minimal compact completion category)**.** A one-family minimal compact completion is a compact internal completion satisfying the following conditions:

(i) it acts on the one-family chiral input of theorem D.1;

(ii) any additional abelian direction is family-universal and commutes with $SU(3)_c \times SU(2)_L$;

(iii) the completion is spectator-free on the one-family carrier;

(iv) the color–lepton carrier is a minimal complex irreducible carrier whose restriction contains one color triplet and one lepton singlet with the Standard Model $B-L$ assignments;

(v) the right weak carrier is the minimal nontrivial compact carrier required to pair the conjugate right sector;

(vi) hypercharge is a linear combination of the right-sector generator and $B-L$;

(vii) the resulting one-family block is anomaly-safe.

Definition D.3 (Chimeric Pati–Salam internal realization)**.** A chimeric Pati–Salam internal realization is a unitary representation

$$\rho_{\mathrm{PS}} : SU(4) \times SU(2)_L \times SU(2)_R \to U(S_{\mathrm{int}})$$

together with a G_{PS}-invariant submodule $W_{\mathrm{fam}} \subset S_{\mathrm{int}}$ whose physical chiral interpretation assigns $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ to left slice chirality and $(\mathbf{4}, \mathbf{1}, \mathbf{2})$ to right slice chirality.

Definition D.4 (One-family Pati–Salam admissible internal module)**.** A physical chiral G_{PS}-module is one-family Pati–Salam admissible if: it contains one left-chiral $SU(2)_L$ weak sector, one right-chiral $SU(2)_R$ weak sector, one $SU(4)$ color–lepton carrier with a right-handed neutrino after branching, hypercharge is linear, the resulting table is anomaly-safe, no spectator states occur, and the dimension is minimal among modules satisfying these conditions.

Lemma D.5 (Family-universal abelian direction theorem)**.** *Let X be a family-universal abelian charge on the one-family* $\mathrm{SM} + \nu_R$ *chiral input, commuting with $SU(3)_c \times SU(2)_L$, and assume the mixed nonabelian, mixed abelian with retained hypercharge, cubic abelian, and mixed gravitational anomaly conditions vanish. Then*

$$X \in \mathrm{span}\{Y, B - L\}.$$

Consequently, modulo the observed hypercharge direction, $B - L$ is the unique new family-universal anomaly-free abelian direction in this category.

Proof. Assign charges

$$X(q_L) = x_q, \quad X(\ell_L) = x_\ell, \quad X(u_R^c) = x_u, \quad X(d_R^c) = x_d, \quad X(e_R^c) = x_e, \quad X(\nu_R^c) = x_\nu.$$

The mixed $SU(3)^2 U(1)_X$ anomaly gives

$$2x_q + x_u + x_d = 0.$$

The mixed $SU(2)_L^2 U(1)_X$ anomaly gives

$$3x_q + x_\ell = 0.$$

Hence

$$x_\ell = -3x_q, \qquad x_d = -2x_q - x_u.$$

The mixed gravitational condition gives

$$6x_q + 2x_\ell + 3x_u + 3x_d + x_e + x_\nu = 0.$$

Substituting the two nonabelian relations gives

$$6x_q + 2(-3x_q) + 3x_u + 3(-2x_q - x_u) + x_e + x_\nu = 0,$$

so

$$x_e + x_\nu = 6x_q.$$

This is the corrected gravitational reduction.

Because hypercharge is retained as a gauge direction, anomaly compatibility of the added abelian direction also requires the mixed abelian coefficient $U(1)_Y^2 U(1)_X$ to vanish. Using the all-left-handed hypercharges, this condition is

$$6\left(\frac{1}{6}\right)^2 x_q + 2\left(-\frac{1}{2}\right)^2 x_\ell + 3\left(-\frac{2}{3}\right)^2 x_u + 3\left(\frac{1}{3}\right)^2 x_d + x_e = 0.$$

Substituting $x_\ell = -3x_q$ and $x_d = -2x_q - x_u$ gives

$$-2x_q + x_u + x_e = 0,$$

hence

$$x_e = 2x_q - x_u.$$

Together with $x_e + x_\nu = 6x_q$, this gives

$$x_\nu = 4x_q + x_u.$$

Thus every compatible charge vector has the form

$$X = (x_q,\, -3x_q,\, x_u,\, -2x_q - x_u,\, 2x_q - x_u,\, 4x_q + x_u)\,.$$

Now solve $X = aY + b(B - L)$. Using the q_L and u_R^c entries,

$$x_q = \frac{a}{6} + \frac{b}{3}, \qquad x_u = -\frac{2a}{3} - \frac{b}{3}.$$

These two equations give

$$a = -2x_q - 2x_u, \qquad b = 4x_q + x_u.$$

Substituting these values into $aY + b(B - L)$ reproduces all six entries above. Therefore

$$X \in \operatorname{span}\{Y, B - L\}.$$

The remaining cubic abelian coefficient and the $U(1)_Y U(1)_X^2$ coefficient vanish on this two-parameter span because the one-family SM+ν_R table is anomaly-free for both Y and $B-L$, including the mixed abelian coefficients. Therefore, after quotienting by the already observed hypercharge direction, the only new family-universal anomaly-free direction is $B - L$. □

Lemma D.6 (Irreducibles of the product group)**.** *Every finite-dimensional irreducible complex representation of $SU(4) \times SU(2)_L \times SU(2)_R$ is an exterior tensor product $R_4 \boxtimes R_L \boxtimes R_R$.*

Proof. Finite-dimensional representations of compact groups are completely reducible, and irreducibles of a direct product are exterior tensor products of irreducibles of the factors. □

Lemma D.7 (Minimal color–lepton $SU(4)$ carrier)**.** *Inside the one-family minimal compact completion category, the anomaly-free $B - L$ direction promotes to the minimal complex color–lepton carrier $SU(4)_C$. Under*

$$SU(4)_C \supset SU(3)_c \times U(1)_{B-L},$$

the minimal irreducible carrier containing exactly one color triplet with $B - L = 1/3$ and one lepton singlet with $B - L = -1$ is $\mathbf{4}$. Its conjugate $\bar{\mathbf{4}}$ contains the left-handed conjugate charges.

Proof. The family-universal abelian direction theorem identifies $B - L$ as the non-hypercharge anomaly-free direction. A minimal complex color–lepton carrier must contain a color triplet and a lepton singlet with the displayed $B - L$ weights. Hence the carrier has complex dimension at least four.

The relevant smaller and competing simple compact possibilities are excluded as follows:

candidate	small carrier	obstruction
$SU(2)$	$\mathbf{2}$	no $SU(3)_c$ subgroup and no color triplet
$SU(3)$	$\mathbf{3}, \bar{\mathbf{3}}$	no independent lepton singlet in the irreducible carrier
$\mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$	$\mathbf{4}$ pseudoreal	does not restrict as $(\mathbf{3})_{1/3} \oplus (\mathbf{1})_{-1}$
G_2	$\mathbf{7}$ real	not a minimal complex four-state color–lepton carrier

A carrier containing $SU(3)_c \times U(1)_{B-L}$ with the required triplet-plus-singlet branching must have rank at least three. Among rank-three simple compact groups, the fundamental of $SU(4)$ realizes the required complex four-state branch:

$$\mathbf{4} \to (\mathbf{3})_{1/3} \oplus (\mathbf{1})_{-1}.$$

The competing rank-three families do not supply a complex four-dimensional irreducible carrier with this restriction: $\mathrm{Sp}(3)$ has pseudoreal fundamental of complex dimension six, and $\mathrm{Spin}(7)$ has real vector and spinor carriers of dimensions seven and eight. Therefore $SU(4)_C$ is the minimal compact color–lepton completion in the declared category. The conjugate fundamental carries the left-handed conjugate charges. □

Lemma D.8 (Minimal right-weak $SU(2)_R$ carrier)**.** *Inside the one-family minimal compact completion category, the right-sector splitting required by hypercharge is carried by the minimal nontrivial compact carrier $SU(2)_R$, with generator T^3_R. The minimal nontrivial weak carrier for the left sector is $(\mathbf{2}, \mathbf{1})$, and the minimal nontrivial weak carrier for the right sector is $(\mathbf{1}, \mathbf{2})$.*

Proof. The right-conjugate sector contains paired states with identical color–lepton charge and different hypercharge. Once $B - L$ is fixed, the remaining splitting in $Y = T^3_R + \frac{1}{2}(B - L)$ is carried by a right-sector generator with opposite eigenvalues. The smallest compact nontrivial carrier with this structure is the $SU(2)$ doublet. The left weak carrier is likewise the fundamental $SU(2)_L$ doublet. Any higher representation or extra direct-sum component introduces additional states and violates the minimal no-spectator condition. □

Lemma D.9 (Minimal chiral weak carriers)**.** *The minimal nontrivial weak carrier for the left sector is $(\mathbf{2}, \mathbf{1})$, and the minimal nontrivial weak carrier for the right sector is $(\mathbf{1}, \mathbf{2})$.*

Proof. This is the weak-carrier part of theorem D.8. The fundamental doublet is the smallest nontrivial $SU(2)$ representation. □

Lemma D.10 (Lower bound for one-family Pati–Salam dimension)**.** *Every one-family Pati–Salam admissible physical chiral module has complex dimension at least* 16*, counted as Weyl states on the slice.*

Proof. The color–lepton carrier contributes at least 4 dimensions. The left weak carrier contributes at least 2, giving 8 left states; the right weak carrier contributes at least 2, giving 8 right states. Hence at least 16. □

Theorem D.11 (Minimal one-family Pati–Salam carrier)**.** *Let S_{int} carry a one-family minimal compact completion in the sense of theorem D.2. Then the selected spectator-free family block is the Pati–Salam carrier*

$$W_{\text{fam}} \simeq (\mathbf{4}, \mathbf{2}, \mathbf{1})_L \oplus (\mathbf{4}, \mathbf{1}, \mathbf{2})_R$$

in physical chiral notation, up to charge conjugation and family replication. In the all-left-handed anomaly convention,

$$W_{\text{fam}}^{\text{LH}} \simeq (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}).$$

Proof. By theorem D.5, the only new family-universal anomaly-free abelian direction modulo hypercharge is $B - L$. By theorem D.7, the minimal color–lepton completion carrying that direction is $SU(4)_C$ on $\mathbf{4}$. By theorem D.8, the minimal right-sector compact carrier is $SU(2)_R$, while the observed left weak carrier is $SU(2)_L$. By theorem D.6, the irreducible modules of the product group are exterior tensor products. The minimal left sector is therefore $(\mathbf{4}, \mathbf{2}, \mathbf{1})_L$, and the minimal right sector is $(\mathbf{4}, \mathbf{1}, \mathbf{2})_R$. Their direct sum saturates the lower bound of theorem D.10. The all-left-handed convention converts the right-handed $\mathbf{4}$ into $\bar{\mathbf{4}}$, while the $SU(2)$ doublet is pseudoreal. □

Scope Note D.12 (Selection domain)**.** The conclusion of theorem D.11 is a selection theorem inside the declared one-family minimal compact completion category. It is not a global uniqueness theorem over every possible internal geometry, every nonminimal extension, every spectator-containing model, or every multi-family construction.

Corollary D.13 (Sixteen Weyl states)**.** *The all-left-handed block contains sixteen left-handed Weyl states.*

Proof. $\dim(\mathbf{4}, \mathbf{2}, \mathbf{1}) = 8$ and $\dim(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) = 8$. □

Proposition D.14 (State-table hypercharge check)**.** *Every entry in the front-matter state table satisfies $Y = T_R^3 + (B - L)/2$.*

Proof. For q_L, $Y = 0 + (1/2)(1/3) = 1/6$. For ℓ_L, $Y = 0 + (1/2)(-1) = -1/2$. For u_R^c, $Y = -1/2 + (1/2)(-1/3) = -2/3$. For d_R^c, $Y = 1/2 + (1/2)(-1/3) = 1/3$. For e_R^c, $Y = 1/2 + (1/2)(1) = 1$. For ν_R^c, $Y = -1/2 + (1/2)(1) = 0$. □

Corollary D.15 (Multiplicity check)**.** *The state table contains sixteen left-handed Weyl states.*

Proof. The multiplicities sum to $6 + 2 + 3 + 3 + 1 + 1 = 16$. □

Export D.16 (One-family state-table export)**.** Paper II exports the all-left-handed one-family state table and the minimal chiral-completion selection chain

$$\mathrm{SM} + \nu_R \Longrightarrow \mathrm{span}\{Y, B - L\} \Longrightarrow SU(4)_C \times SU(2)_L \times SU(2)_R \Longrightarrow W^{\mathrm{LH}}_{\mathrm{fam}}.$$

GU III uses this table for anomaly traces, and GU IV uses it for current and degeneracy bookkeeping unless a new state ledger is opened.

E Pati–Salam Branching, Hypercharge Uniqueness, and Spin(10) Envelope

Embed $SU(3)_c$ as the upper-left block in $SU(4)$. Let

$$T_{15} = \frac{1}{2\sqrt{6}} \operatorname{diag}(1, 1, 1, -3), \qquad \operatorname{tr}(T_{15}^2) = \frac{1}{2}.$$

The physical generator is

$$B - L = 2\sqrt{\frac{2}{3}} T_{15}, \qquad Q_{BL} = \frac{1}{2}(B - L) = \sqrt{\frac{2}{3}} T_{15}.$$

Lemma E.1 ($SU(4)$ color–lepton branching)**.** *Under $SU(4) \supset SU(3)_c \times U(1)_{B-L}$,*

$$\mathbf{4} \to (\mathbf{3})_{1/3} \oplus (\mathbf{1})_{-1}, \qquad \bar{\mathbf{4}} \to (\bar{\mathbf{3}})_{-1/3} \oplus (\mathbf{1})_{+1}.$$

Proof. Multiplying T_{15} by $2\sqrt{2/3}$ gives $(1/3) \operatorname{diag}(1, 1, 1, -3)$. Thus the fundamental has charges $(1/3, 1/3, 1/3, -1)$; the antifundamental has conjugate color representation and opposite charge. □

Theorem E.2 (Hypercharge uniqueness)**.** *Assume the $SU(4)$ branching and the all-left-handed family submodule. If*

$$Y = aT_R^3 + b(B - L)$$

and the q_L and u_R^c entries have their displayed Standard Model hypercharges, then

$$a = 1, \qquad b = \frac{1}{2},$$

so

$$Y = T_R^3 + \frac{1}{2}(B - L).$$

Proof. For q_L, $T_R^3 = 0$, $B - L = 1/3$, and $Y = 1/6$, so $b = 1/2$. For u_R^c, $T_R^3 = -1/2$, $B - L = -1/3$, and $Y = -2/3$; substituting $b = 1/2$ gives $-a/2 - 1/6 = -2/3$, hence $a = 1$. □

Corollary E.3 (Hypercharge table consistency)**.** *The hypercharge map reproduces every row in the state table.*

Proof. This is theorem D.14. □

Theorem E.4 (Pati–Salam one-family branching)**.** *The family submodule $W_{\mathrm{fam}}^{\mathrm{LH}}$ branches to the all-left-handed Standard Model one-family state table.*

Proof. The left block branches as $(\mathbf{4}, \mathbf{2}, \mathbf{1}) \to (\mathbf{3}, \mathbf{2})_{1/6} \oplus (\mathbf{1}, \mathbf{2})_{-1/2}$. The right-conjugate block branches as $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) \to (\bar{\mathbf{3}}, \mathbf{1})_{1/3} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-2/3} \oplus (\mathbf{1}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{1})_0$. These are the displayed state-table rows. □

Definition E.5 (Minimal-rank simple chiral envelope problem)**.** The minimal-rank simple chiral envelope problem asks for a compact simple group G and a complex chiral representation R such that:

(i) $G_{\text{PS}} = SU(4)_C \times SU(2)_L \times SU(2)_R$ embeds in G;

(ii) R restricts to the one-family Pati–Salam carrier $W^{\text{LH}}_{\text{fam}}$;

(iii) the rank is minimal among simple compact groups satisfying the first two conditions;

(iv) the representation is chiral and spectator-free on restriction to G_{PS}.

Theorem E.6 (Minimal-rank Spin(10) chiral envelope)**.** *Inside the minimal-rank simple chiral envelope problem of theorem E.5, the one-family Pati–Salam carrier embeds in the chiral spinor* **16** *of* Spin(10)*, with branching*

$$\mathbf{16}\downarrow_{SU(4)_C\times SU(2)_L\times SU(2)_R} = (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}).$$

Proof. The Pati–Salam group has rank

$$\operatorname{rank}(SU(4)_C) + \operatorname{rank}(SU(2)_L) + \operatorname{rank}(SU(2)_R) = 3 + 1 + 1 = 5.$$

A simple compact envelope containing it must therefore have rank at least five. The rank-five simple compact Lie algebras are A_5, B_5, C_5, and D_5, corresponding to

$$SU(6), \qquad \mathrm{Spin}(11), \qquad \mathrm{Sp}(5), \qquad \mathrm{Spin}(10).$$

The required representation must be complex, chiral, spectator-free on restriction to G_{PS}, and must restrict to

$$(\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}).$$

The rank-five sweep is:

Rank-five group	**Relevant small representations**	**Obstruction/status**
$SU(6)$	$\mathbf{6}, \mathbf{15}, \mathbf{20}$	No spectator-free complex **16** restricting to $W^{\text{LH}}_{\text{fam}}$.
Sp(5)	**10** pseudoreal and higher symplectic carriers	Pseudoreal/symplectic carrier, not the complex chiral **16**.
Spin(11)	$\mathbf{11}, \mathbf{32}$	Spinor restricts through D_5 as a doubled $\mathbf{16}\oplus\overline{\mathbf{16}}$ sector.
Spin(10)	$\mathbf{16}_+, \mathbf{16}_-$	Chiral spinor restricts exactly to $W^{\text{LH}}_{\text{fam}}$.

For $D_5 = \mathrm{Spin}(10)$, the standard subgroup

$$\mathrm{Spin}(10) \supset \mathrm{Spin}(6) \times \mathrm{Spin}(4) \simeq SU(4)_C \times SU(2)_L \times SU(2)_R$$

gives the chiral spinor branching

$$\mathbf{16}\downarrow_{G_{\mathrm{PS}}} = (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}).$$

This realizes the desired carrier at minimal rank and without spectator states in the restriction. Therefore Spin(10) supplies the minimal-rank simple chiral envelope in the declared category. □

Corollary E.7 (Carrier chain to the axial current)**.** *The GU II matter carrier chain is*

$$\mathrm{SM} + \nu_R \Longrightarrow \mathrm{span}\{Y, B-L\} \Longrightarrow SU(4)_C \times SU(2)_L \times SU(2)_R \Longrightarrow \mathbf{16}_{\mathrm{Spin}(10)} \Longrightarrow J_5^\mu.$$

The final arrow is the internal-singlet Lorentz axial-current construction of theorems A.5 and A.6.

Proof. The first arrow is theorem D.5. The second and third arrows are theorems D.7, D.8 and D.11. The Spin(10) envelope is theorem E.6. The current bridge is theorems A.5 and A.6. □

Export E.8 (Pati–Salam, hypercharge, and Spin(10) envelope export)**.** Paper II exports

$$W_{\mathrm{fam}}^{\mathrm{LH}} \to q_L \oplus \ell_L \oplus u_R^c \oplus d_R^c \oplus e_R^c \oplus \nu_R^c,$$

with

$$Y = T_R^3 + \frac{1}{2}(B-L),$$

and records the minimal-rank simple chiral envelope

$$\mathbf{16}_{\mathrm{Spin}(10)}\downarrow_{G_{\mathrm{PS}}} = (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}).$$

This export is read by GU III for anomaly and BRST/BV bookkeeping and by GU IV for current/-carrier accounting. The envelope theorem does not add new family states, and it does not assert a scalar potential, mass hierarchy, or three-family replication theorem.

F Anomaly Sums and Index Cross-Check

The anomaly conventions follow standard gauge-theory treatments [8–13]. The one-family minimal chiral-completion chain imported into Paper II is recorded in [20]; this appendix supplies the explicit anomaly ledger used by the GU II matter-current export.

Theorem F.1 (One-family perturbative anomaly cancellation)**.** *For the state table of $W^{\mathrm{LH}}_{\mathrm{fam}}$, the perturbative anomalies*

$$SU(3)^3, \quad SU(3)^2U(1)_Y, \quad SU(2)^2_L U(1)_Y, \quad U(1)^3_Y, \quad \mathrm{grav}^2 U(1)_Y$$

vanish per generation.

Proof. For $SU(3)^3$, $2A(\mathbf{3}) + A(\bar{\mathbf{3}}) + A(\bar{\mathbf{3}}) = 2 - 1 - 1 = 0$. For $SU(3)^2U(1)_Y$,

$$2(1/6)(1/2) + (-2/3)(1/2) + (1/3)(1/2) = 0.$$

For $SU(2)^2_L U(1)_Y$,

$$3(1/6)(1/2) + (-1/2)(1/2) = 0.$$

For $U(1)^3_Y$,

$$6(1/6)^3 + 3(-2/3)^3 + 3(1/3)^3 + 2(-1/2)^3 + 1^3 + 0^3 = 0.$$

For the mixed gravitational anomaly,

$$6(1/6) + 3(-2/3) + 3(1/3) + 2(-1/2) + 1 + 0 = 0.$$

□

Theorem F.2 (Witten $SU(2)_L$ anomaly check)**.** *The one-family state table has no global $SU(2)_L$ Witten anomaly.*

Proof. There are three colored quark doublets and one lepton doublet, so $N_{\mathrm{doublets}} = 3 + 1 = 4$, which is even [13]. □

Theorem F.3 (Full $B - L$ anomaly ledger with ν^c_R)**.** *The one-family state table including ν^c_R cancels $SU(3)^2U(1)_{B-L}$, $SU(2)^2_L U(1)_{B-L}$, $U(1)^3_{B-L}$, and* $\mathrm{grav}^2 U(1)_{B-L}$*. If $SU(2)_R \times U(1)_{B-L}$ is retained, $SU(2)^2_R U(1)_{B-L}$ also vanishes.*

Proof. The sums are

$$2(1/3)(1/2) + (-1/3)(1/2) + (-1/3)(1/2) = 0,$$

$$3(1/3)(1/2) + (-1)(1/2) = 0,$$

$$6(1/3)^3 + 3(-1/3)^3 + 3(-1/3)^3 + 2(-1)^3 + 1^3 + 1^3 = 0,$$

and

$$6(1/3) + 3(-1/3) + 3(-1/3) + 2(-1) + 1 + 1 = 0.$$

For $SU(2)_R^2(B-L)$, $3(-1/3)(1/2) + 1(1)(1/2) = 0$. □

Corollary F.4 ($B-L$ is available as the anomaly-safe color–lepton direction)**.** *Inside the one-family minimal compact completion category, the $B-L$ direction is not an additional anomaly obstruction. With ν_R^c included, it is the anomaly-safe family-universal direction consumed by the $SU(4)_C$ color–lepton completion.*

Proof. The family-universal abelian direction theorem identifies $B-L$ as the non-hypercharge anomaly-free direction in the declared category. The explicit $B-L$ anomaly ledger of theorem F.3 verifies that the direction closes on the all-left-handed one-family table with ν_R^c. □

Theorem F.5 (Pati–Salam block anomaly check)**.** *The Pati–Salam family submodule is perturbatively anomaly-free for nonabelian cubic anomalies and has no global $SU(2)$ anomaly in either $SU(2)$ factor.*

Proof. For $SU(4)^3$, $2A(\mathbf{4}) + 2A(\bar{\mathbf{4}}) = 2 - 2 = 0$. The $SU(2)$ local cubic anomalies vanish. The global $SU(2)$ checks have four doublets in each $SU(2)$ factor, hence even. □

Theorem F.6 (Spin(10) envelope anomaly compatibility)**.** *The* **16** *chiral envelope of* Spin(10)*, restricted to*

$$SU(4)_C \times SU(2)_L \times SU(2)_R,$$

has the anomaly ledger of the one-family Pati–Salam block:

$$\mathbf{16}\downarrow_{G_{\mathrm{PS}}} = (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}).$$

Consequently, the Spin(10) *envelope introduces no additional one-family spectator contribution to the GU II anomaly ledger.*

Proof. The minimal-rank envelope theorem gives the displayed branching. Since the restriction is exactly the Pati–Salam family block, with no additional irreducible spectator component, the anomaly sums reduce to theorem F.5 and the Standard Model ledger of theorems F.1 to F.3. □

Theorem F.7 (Anomaly-polynomial vanishing)**.** *The total six-form anomaly polynomial of one Paper II family vanishes:*

$$I_6^{\text{one family}} = 0.$$

Proof. The degree-six coefficients in

$$I_6(\psi_L) = \left[\widehat{A}(TX)\, \mathrm{ch}_R\left(\frac{\mathcal{F}}{2\pi}\right) e^{qx}\right]_6$$

are the nonabelian cubic, mixed nonabelian–abelian, cubic abelian, and mixed gravitational–abelian anomaly coefficients computed above. Each vanishes, so the sum over $W_{\mathrm{fam}}^{\mathrm{LH}}$ is zero. □

Export F.8 (Anomaly export)**.** Paper II exports the full one-family anomaly ledger for $W_{\mathrm{fam}}^{\mathrm{LH}}$: Standard Model perturbative anomalies vanish, the global $SU(2)_L$ anomaly is absent, the $B - L$ ledger closes with ν_R^c, the Pati–Salam block is locally and globally anomaly-safe, the Spin(10) envelope restricts without spectator anomaly content, and $I_6 = 0$.

G Gauge-Admissible Scalar and Yukawa Seed Module

This appendix proves selection rules, representation channels, and singlet existence. It does not prove numerical mass textures, scalar-potential stability, vacuum existence, or family hierarchies.

Theorem G.1 (Yukawa selection rule)**.** *A Yukawa seed $\psi_a\psi_b\phi$ is gauge-allowed if and only if*

$$R_a \otimes R_b \otimes R_\phi \supset \mathbf{1}.$$

If the trivial representation is absent, the corresponding gauge-invariant seed vanishes.

Proof. A term in the action must be invariant under the gauge group. A nonzero invariant linear functional on $R_a \otimes R_b \otimes R_\phi$ exists precisely when the tensor product contains the trivial representation. Otherwise group averaging projects the candidate to zero. □

Theorem G.2 (Pati–Salam Yukawa scalar representations)**.** *The fermion bilinear transforms as*

$$(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}) \otimes (\mathbf{4}, \mathbf{1}, \mathbf{2}) = (\mathbf{1}, \mathbf{2}, \mathbf{2}) \oplus (\mathbf{15}, \mathbf{2}, \mathbf{2}).$$

Consequently, the minimal scalar channel is $(\mathbf{1}, \mathbf{2}, \mathbf{2})$, with optional $(\mathbf{15}, \mathbf{2}, \mathbf{2})$.

Proof. $\bar{\mathbf{4}} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{15}$, while $\mathbf{2}_L \otimes \mathbf{1}_L = \mathbf{2}_L$ and $\mathbf{1}_R \otimes \mathbf{2}_R = \mathbf{2}_R$. The stated decomposition follows. □

Proposition G.3 (Bi-doublet singlet contraction)**.** *The contraction*

$$\overline{\Psi}_{L\,\alpha i}\, {\Phi_H}^{i}{}_{\dot{j}}\, \Psi_R^{\alpha \dot{j}}$$

is a Pati–Salam singlet.

Proof. The $SU(4)$ index α, $SU(2)_L$ index i, and $SU(2)_R$ index $\dot{j}$ are each contracted with their dual partners. □

Corollary G.4 (Spin(10) carrier does not change the minimal Yukawa seed ledger)**.** *Passing from the Pati–Salam family block to its minimal-rank* Spin(10) *envelope does not alter the GU II minimal slice Yukawa seed ledger. The GU II minimal seed channel remains the Pati–Salam bi-doublet channel*

$$(\mathbf{1}, \mathbf{2}, \mathbf{2}),$$

with $(\mathbf{15}, \mathbf{2}, \mathbf{2})$ retained as a nonminimal Pati–Salam channel.

Proof. The Spin(10) envelope theorem supplies a simple chiral carrier whose restriction to G_{PS} is exactly $W^{\mathrm{LH}}_{\mathrm{fam}}$. The GU II Yukawa seed ledger is a representation statement on the restricted Pati–Salam family block. Therefore the allowed Pati–Salam scalar channels are the ones computed in theorem G.2. Additional Spin(10)-level scalar representations require a separate scalar-module declaration and do not alter the minimal GU II seed ledger. □

Theorem G.5 (Gauge-admissible scalar-channel criterion)**.** *Let an internal mode ϕ define an operator $\mathcal{O}_\phi$ on W_{fam}. If $\mathcal{O}_\phi$ transforms as $(\mathbf{1}, \mathbf{2}, \mathbf{2})$ or $(\mathbf{15}, \mathbf{2}, \mathbf{2})$, then the corresponding overlap is gauge-allowed whenever the singlet contraction is nonzero. If $\mathcal{O}_\phi$ transforms in no representation pairing with $\overline{\Psi}_L \Psi_R$ to form a singlet, the overlap vanishes by gauge invariance.*

Proof. Combine theorems G.1 and G.2. ☐

Export G.6 (Yukawa seed export)**.** Paper II exports the gauge-invariant Yukawa selection rule, the bilinear decomposition $(\mathbf{1}, \mathbf{2}, \mathbf{2}) \oplus (\mathbf{15}, \mathbf{2}, \mathbf{2})$, and the minimal Pati–Salam scalar channel $(\mathbf{1}, \mathbf{2}, \mathbf{2})$. Numerical mass textures, scalar-potential stability, vacuum selection, geometric scalar existence, and family hierarchies require additional data.

H Gauge Normalization and Mixing Ledger

The hypercharge generator is

$$Y = T_R^3 + \frac{1}{2}(B - L).$$

Let $Q_{BL} = (B - L)/2 = \sqrt{2/3}T_{15}$.

Proposition H.1 (B − L coupling normalization)**.** *If g_4 is the $SU(4)$ coupling normalized with* $\mathrm{tr}(T_{15}^2) = 1/2$*, then*

$$g_{BL} = \sqrt{\frac{3}{2}}g_4.$$

Proof. The $SU(4)$ interaction along the T_{15} direction is $g_4 T_{15} A_\mu^{15}$. Writing the same interaction with $Q_{BL} = \sqrt{2/3}T_{15}$ gives $g_{BL} Q_{BL} A_\mu^{BL}$. Hence $g_{BL}\sqrt{2/3} = g_4$. □

Theorem H.2 (Massless hypercharge kernel)**.** *Suppose a scalar vacuum breaks $SU(2)_R \times U(1)_{B-L}$ while preserving $Y = T_R^3 + Q_{BL}$. Then the neutral gauge-boson mass matrix has a one-dimensional kernel generated by*

$$B_\mu^Y = \frac{g_Y}{g_R} A_\mu^R + \frac{g_Y}{g_{BL}} A_\mu^{BL},$$

where

$$\frac{1}{g_Y^2} = \frac{1}{g_R^2} + \frac{1}{g_{BL}^2}.$$

Proof. A vacuum neutral under Y has charges $t_R + q_{BL} = 0$. The mass term is proportional to $(g_R A_\mu^R - g_{BL} A_\mu^{BL})^2$, whose kernel is generated by the displayed orthogonal combination. Canonical normalization yields the coupling relation. □

Theorem H.3 (Pati–Salam unification normalization)**.** *If $g_R = g_4 = g$ at a Pati–Salam unification scale, then*

$$g_Y^2 = \frac{3}{5}g^2, \qquad g_1^2 = \frac{5}{3}g_Y^2 = g^2.$$

Proof. By theorem H.1, $g_{BL}^2 = 3g^2/2$. Thus $1/g_Y^2 = 1/g^2 + 2/(3g^2) = 5/(3g^2)$. □

Scope Boundary H.4 (Gauge-normalization scope)*.* The normalization ledger fixes generator normalization, hypercharge embedding, and the tree-level kernel relation for the declared Pati–Salam breaking pattern. It does not assert a unification scale, threshold-correction model, scalar potential, or measured coupling fit.

Export H.5 (Gauge-normalization export)**.** Paper II exports $Y = T_R^3 + (B-L)/2$, $Q_{BL} = \sqrt{2/3}T_{15}$, $g_{BL} = \sqrt{3/2}g_4$, and $1/g_Y^2 = 1/g_R^2 + 1/g_{BL}^2$.

I Current and Thermal Ledgers

The thermal formulas follow standard cosmology and thermal-field-theory conventions [14–17]; the current variation follows the conserved-current variational principle for relativistic fluids [18, 19].

Definition I.1 (Axial conservation class)**.** Paper II distinguishes:

$$\nabla_\mu J_5^\mu = 0, \qquad \nabla_\mu J_5^\mu = \mathcal{S}_{\rm anom}, \qquad \nabla_\mu J_5^\mu = \mathcal{S}_{\rm source}.$$

GU IV may use $n_5(a) = n_{5,0}a^{-3}$ only in the conserved class or after an effective conserved branch has been declared.

Domain Condition I.2 (Conserved-branch source ledger)**.** A generic axial current obeys the schematic ledger

$$\nabla_\mu J_5^\mu = \sum_i c_i \frac{g_i^2}{16\pi^2} F_i \widetilde{F}_i + 2im\bar{\psi}\gamma^5\psi + \mathcal{S}_{\rm int}.$$

The conserved branch is the domain where these terms vanish or are negligible on the interval of use.

Lemma I.3 (Homogeneous conserved scaling)**.** *If $\nabla_\mu J_5^\mu = 0$ and $J_5^\mu = n_5 u^\mu$ in FLRW, then*

$$n_5(a) = n_{5,0}a^{-3}.$$

Proof. In FLRW, $\nabla_\mu(n_5u^\mu) = \dot{n}_5 + 3Hn_5$. Conservation gives $d(a^3n_5)/dt = 0$. □

Proposition I.4 (Sourced axial-current evolution)**.** *If $\nabla_\mu J_5^\mu = \mathcal{S}(t)$, then*

$$\frac{d}{dt}(a^3n_5) = a^3\mathcal{S}(t),$$

and

$$n_5(t) = a(t)^{-3}\left[a(t_i)^3 n_5(t_i) + \int_{t_i}^{t} a(t')^3\mathcal{S}(t')dt'\right].$$

Proof. Use $\nabla_\mu(n_5u^\mu) = \dot{n}_5 + 3Hn_5$ and multiply by a^3. □

Lemma I.5 (Relativistic thermal reference densities)**.** *For relativistic species,*

$$\rho(T) = \frac{\pi^2}{30}g_*(T)T^4, \qquad n_f(T) = \frac{3\zeta(3)}{4\pi^2}g_fT^3,$$

and for $|\mu| \ll T$,

$$\Delta n_f = \frac{g_f}{6}\mu T^2 + O(\mu^3).$$

Proof. These follow from the standard Bose–Einstein and Fermi–Dirac integrals $\int_0^\infty x^3/(e^x-1)dx = \pi^4/15$, $\int_0^\infty x^3/(e^x+1)dx = 7\pi^4/120$, and $\int_0^\infty x^2/(e^x+1)dx = 3\zeta(3)/2$, plus the small-$\mu/T$ expansion of the Fermi distribution. □

Definition I.6 (Axial population weights)**.** Let species i have degeneracy g_i and axial-current weight w_i. Define

$$g_{5,\text{eff}} = \sum_i w_i g_i.$$

Theorem I.7 (Thermal axial reference density requires an axial branch)**.** *If a thermal state populates an axial branch with declared weights w_i, then*

$$n_{5,\text{ref}}(T) = \frac{3\zeta(3)}{4\pi^2} g_{5,\text{eff}} T^3.$$

If the weighted sum vanishes, the axial reference density vanishes even when the total thermal density is nonzero.

Proof. Sum the fermionic number densities with weights w_i: $n_5 = \sum_i w_i n_i$. □

I.1 Sterile dense-fermion branch

The conserved homogeneous branch and the thermal reference branch do not exhaust the admissible matter-current data that may be declared downstream. Paper II also records a species-resolved weak-singlet source class, called the sterile dense-fermion branch. This is a matter/current source class. It is not a GU III legality certificate, not a parent-source calculation, not a tensor-amplitude theorem, and not a GU IV observable map.

Definition I.8 (Sterile dense-fermion branch)**.** A sterile dense-fermion branch is a species-resolved occupied Dirac sector satisfying the following matter-current gates on the interval of use:

$$T_{\text{spin}} \ll \bar{\mu}, \qquad \frac{\mu_5}{\bar{\mu}} \gtrsim 0.27, \qquad \Gamma_\chi \tau_{\text{prod}} \lesssim 1, \qquad \rho_{\text{deg}} \lesssim \rho_b.$$

The carrier is weak-singlet/sterile or otherwise sphaleron-blind on the production interval. The branch supplies declared source-class data for downstream species-resolution; it does not by itself certify anomaly, washout, RG, or sign-corridor legality.

Domain Condition I.9 (Sterile dense-fermion branch import domain)**.** A downstream layer may consume the sterile dense-fermion branch only with a declared occupied species list, degeneracy g_p, charge/gauge class, chemical-potential split, chirality-flip ledger, production interval, and energy-budget ledger. The energy-budget ledger must certify the admissibility gate $\rho_{\text{deg}} \lesssim \rho_b$. A packet failing that gate is a control or rejected source packet, not an admissible sterile dense-fermion branch. GU III must certify legality before GU IV or any source-adapter calculation uses the branch as an active packet.

Definition I.10 (Degenerate axial polarization)**.** For a degenerate branch with Fermi momenta p_{F+} and p_{F-}, define

$$n_\pm = \frac{g_p}{6\pi^2} p_{F\pm}^3, \qquad P_A = \frac{n_+ - n_-}{n_+ + n_-}.$$

Theorem I.11 (Degenerate Fermi-surface polarization)**.** *For the degenerate axial polarization of theorem I.10,*

$$P_A = \frac{p_{F+}^3 - p_{F-}^3}{p_{F+}^3 + p_{F-}^3}.$$

If

$$r = \frac{p_{F+}}{p_{F-}},$$

then

$$P_A(r) = \frac{r^3 - 1}{r^3 + 1}.$$

The threshold $P_A \gtrsim 0.68$ is achieved for

$$r \gtrsim 1.74.$$

Proof. Substituting

$$n_\pm = \frac{g_p}{6\pi^2} p_{F\pm}^3$$

into

$$P_A = \frac{n_+ - n_-}{n_+ + n_-}$$

cancels the common factor $g_p/(6\pi^2)$ and gives

$$P_A = \frac{p_{F+}^3 - p_{F-}^3}{p_{F+}^3 + p_{F-}^3}.$$

Writing $p_{F+} = r p_{F-}$ gives

$$P_A(r) = \frac{r^3 p_{F-}^3 - p_{F-}^3}{r^3 p_{F-}^3 + p_{F-}^3} = \frac{r^3 - 1}{r^3 + 1}.$$

Solving $P_A(r) \geq 0.68$ gives

$$\frac{r^3 - 1}{r^3 + 1} \geq 0.68 \quad \Longleftrightarrow \quad r^3 \geq \frac{1 + 0.68}{1 - 0.68} = 5.25.$$

Thus $r \geq 5.25^{1/3} \simeq 1.74$. □

Lemma I.12 (Chemical-potential split and occupation ratio)**.** *For a relativistic degenerate branch with $\mu_\pm \simeq p_{F\pm}$, define*

$$\mu_\pm = \bar{\mu}(1 \pm \delta), \qquad \delta = \frac{\mu_5}{\bar{\mu}}.$$

Then

$$r = \frac{p_{F+}}{p_{F-}} \simeq \frac{1 + \delta}{1 - \delta}.$$

For $\delta \simeq 0.27$,

$$r \simeq 1.74, \qquad P_A \simeq 0.68.$$

Proof. In the relativistic degenerate limit, the Fermi momenta are approximated by the chemical potentials. Therefore

$$r \simeq \frac{\mu_+}{\mu_-} = \frac{1+\delta}{1-\delta}.$$

At $\delta = 0.27$, this gives $r = 1.27/0.73 \simeq 1.74$. The corresponding polarization follows from theorem I.11. □

Theorem I.13 (Sterile-sector energy admissibility). *For a relativistic degenerate sterile sector,*

$$\rho_{\rm deg} = \frac{g_p}{8\pi^2}\left(\mu_+^4 + \mu_-^4\right).$$

With

$$\mu_\pm = \bar{\mu}(1 \pm \delta), \qquad \delta \simeq 0.27,$$

one has

$$(1+\delta)^4 + (1-\delta)^4 \simeq 2.88.$$

Compared with

$$\rho_b = \frac{\pi^2}{30} g_* T_b^4,$$

the ratio is

$$\frac{\rho_{\rm deg}}{\rho_b} \simeq 0.111 \frac{g_p}{g_*}\left(\frac{\bar{\mu}}{T_b}\right)^4.$$

For $g_p = 2$ and $g_* = 100$,

$$\frac{\rho_{\rm deg}}{\rho_b} \simeq 0.00222 \left(\frac{\bar{\mu}}{T_b}\right)^4.$$

Thus a small sterile sector with $\bar{\mu} \sim 3T_b$–$4T_b$ is energy-admissible, while $\bar{\mu} \sim 5T_b$ is outside this representative budget.

Proof. Substitute

$$\mu_\pm = \bar{\mu}(1 \pm \delta)$$

into the degenerate energy density:

$$\rho_{\rm deg} = \frac{g_p}{8\pi^2}\bar{\mu}^4\left[(1+\delta)^4 + (1-\delta)^4\right].$$

At $\delta = 0.27$,

$$(1+\delta)^4 + (1-\delta)^4 = 1.27^4 + 0.73^4 \simeq 2.88.$$

Therefore

$$\rho_{\rm deg} \simeq \frac{2.88\, g_p}{8\pi^2}\bar{\mu}^4.$$

Dividing by

$$\rho_b = \frac{\pi^2}{30} g_* T_b^4$$

gives

$$\frac{\rho_{\rm deg}}{\rho_b} \simeq \frac{2.88 g_p}{8\pi^2} \frac{30}{\pi^2 g_*} \left(\frac{\bar{\mu}}{T_b}\right)^4 \simeq 0.111 \frac{g_p}{g_*} \left(\frac{\bar{\mu}}{T_b}\right)^4 .$$

For $g_p = 2$, $g_* = 100$, this becomes

$$0.00222 \left(\frac{\bar{\mu}}{T_b}\right)^4 .$$

The sample values are

$$\bar{\mu} = 3T_b \quad \Rightarrow \quad \rho_{\rm deg}/\rho_b \simeq 0.18,$$

$$\bar{\mu} = 4T_b \quad \Rightarrow \quad \rho_{\rm deg}/\rho_b \simeq 0.57,$$

and

$$\bar{\mu} = 5T_b \quad \Rightarrow \quad \rho_{\rm deg}/\rho_b \simeq 1.39.$$

This proves the representative admissibility window and the corresponding budget boundary. □

Corollary I.14 (Energy gate as an admissibility filter)**.** *The condition*

$$\rho_{\rm deg} \lesssim \rho_b$$

is an admissibility gate for the sterile dense-fermion branch, not an observational preference. A source packet whose degenerate-sector energy exceeds the available bounce budget is outside the branch, even if its formal axial polarization and current algebra are otherwise well-defined.

Proof. The branch definition in theorem I.8 includes $\rho_{\rm deg} \lesssim \rho_b$ as one of its four gates. Theorem I.13 gives the representative energy ratio and exhibits the budget boundary. Therefore any packet with $\rho_{\rm deg}/\rho_b$ above the declared admissible budget fails the branch definition itself. □

Domain Condition I.15 (Washout survival gate)**.** The sterile dense-fermion branch requires

$$\Gamma_\chi \tau_{\rm prod} \lesssim 1.$$

For a mass-suppressed chirality or spin-flip rate of the form

$$\Gamma_\chi \sim \left(\frac{m_{\rm eff}}{\bar{\mu}}\right)^2 \Gamma_{\rm scatt},$$

a sufficient survival condition is

$$\left(\frac{m_{\rm eff}}{\bar{\mu}}\right)^2 \Gamma_{\rm scatt} \tau_{\rm prod} \lesssim 1.$$

Equivalently,

$$\frac{m_{\rm eff}}{\bar{\mu}} \lesssim (\Gamma_{\rm scatt} \tau_{\rm prod})^{-1/2} .$$

Proof. Substitute the rate estimate for Γ_χ into the survival requirement:

$$\Gamma_\chi \tau_{\rm prod} \sim \left(\frac{m_{\rm eff}}{\bar{\mu}}\right)^2 \Gamma_{\rm scatt} \tau_{\rm prod} \lesssim 1.$$

Taking square roots gives the displayed sufficient condition. □

Scope Boundary I.16 (GU II branch status). The sterile dense-fermion branch is a GU II matter/current source class. GU II defines its occupation, polarization, energy, and washout gates. GU III certifies anomaly, washout, BRST/BV, boundary, and sign-corridor legality. GU IV maps declared source-adapter packets to observables only after the branch gates have been declared. GU V audits the declared packet, its domain, and its observable-interface status. GU II does not compute a universal parent-collapse value, tensor amplitude, detector corridor, or observational fit.

Export I.17 (Sterile dense-fermion branch export)**.** Paper II exports the sterile dense-fermion branch as a source-class packet:

$$\text{GU2-K} = \left\{ \begin{matrix} \text{occupied species list}, & \text{weak-singlet/sterile carrier status}, & T_{\text{spin}}, & \bar{\mu}, \\ \mu_5, & P_A(r), & \Gamma_\chi \tau_{\text{prod}}, & \rho_{\text{deg}}/\rho_b \end{matrix} \right\}.$$

This packet is source-class data. It is not a universal value of σ_0, not a detector amplitude, not an observable map, and not a legality certificate without GU III. A packet failing the washout gate or the energy gate $\rho_{\text{deg}} \lesssim \rho_b$ is not an admissible instance of GU2-K.

I.2 Conserved-branch stress tensor

Lemma I.18 (Metric variation of the current magnitude)**.** *At fixed densitized current $\mathcal{J}^\mu = \sqrt{-g}nu^\mu$,*

$$\delta n = \frac{1}{2} n(g_{\mu\nu} + u_\mu u_\nu)\delta g^{\mu\nu}.$$

Proof. This is the standard fixed-densitized-current variation identity. It follows by writing $n^2 = -g_{\mu\nu}\mathcal{J}^\mu\mathcal{J}^\nu/(-g)$ and varying at fixed $\mathcal{J}^\mu$. □

Theorem I.19 (Negative-stiff stress tensor from a quadratic conserved current)**.** *Let*

$$S_f = \int_X \sqrt{-g}\, f(n) d^4x, \qquad f(n) = C_{55} n^2, \qquad C_{55} > 0,$$

with the conserved densitized current held fixed in metric variation. Then

$$\rho = -f(n), \qquad p = f(n) - nf'(n).$$

For $f(n) = C_{55}n^2$,

$$\rho = p = -C_{55}n^2, \qquad w = 1.$$

Proof. Using $\delta\sqrt{-g} = -(1/2)\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ and theorem I.18,

$$\delta S_f = \frac{1}{2} \int \sqrt{-g} \left[-fg_{\mu\nu} + nf'(n)(g_{\mu\nu} + u_\mu u_\nu) \right] \delta g^{\mu\nu} d^4x.$$

Therefore $T_{\mu\nu} = (f - nf')g_{\mu\nu} - nf' u_\mu u_\nu$. Comparing with $(\rho + p)u_\mu u_\nu + pg_{\mu\nu}$ gives $p = f - nf'$, $\rho = -f$. For $f = C_{55}n^2$, $f' = 2C_{55}n$, so $p = \rho = -C_{55}n^2$. □

Corollary I.20 (σ_0 current handoff). *If the conserved branch has* $n_5(a) = n_{5,0}a^{-3}$ *and* $\sigma_0^2 = C_{55}n_{5,0}^2$, *then*

$$\rho_5(a) = p_5(a) = -\sigma_0^2 a^{-6}, \qquad w_5 = 1.$$

Proof. Substitute $n_5(a) = n_{5,0}a^{-3}$ into $\rho_5 = p_5 = -C_{55}n_5^2$. □

Export I.21 (Current and thermal export). Paper II exports J^μ, J_5^μ, $n_5 = -u_\mu J_5^\mu$, conservation/-source classes, the spin-density convention, thermal reference scalings, axial-population weights, and the conserved-branch handoff

$$\sigma_0^2 = C_{55}n_{5,0}^2, \qquad \rho_5 = p_5 = -\sigma_0^2 a^{-6}.$$

It also exports the sterile dense-fermion branch packet GU2-K as a species-resolved source-class packet. GU IV may use $n_5(a) = n_{5,0}a^{-3}$ only when the conserved class or an effective conserved branch is declared; downstream source-adapter use of the sterile dense-fermion branch requires GU III legality status, a declared species-resolution packet, and satisfaction of the washout and energy-admissibility gates.

J Artifact, Reproducibility, and Downstream Export Checklist

This appendix states the reproducibility contract associated with the proof spine.

Definition J.1 (Paper II artifact set)**.** A complete Paper II release requires the following artifacts if artifact release is claimed:

1. `state_table.csv`;
2. `anomaly_ledger.csv` or `anomaly_ledger.xlsx`;
3. `pati_salam_branching.yaml` or equivalent table;
4. `gauge_normalization.yaml`;
5. `current_ledger.yaml`;
6. `minimal_chiral_completion_ledger.yaml`;
7. `spin10_envelope_ledger.yaml`;
8. `sterile_dense_fermion_branch.yaml`;
9. `branch_gate_ledger.yaml`;
10. `family_submodule_manifest.yaml`;
11. `export_manifest.yaml`.

Each artifact records filename, version, date, checksum, and proof label.

Definition J.2 (Artifact checksum)**.** For an artifact file A, its checksum is the cryptographic hash $\mathrm{hash}(A)$, computed by the repository's declared hash function, for example SHA-256.

Definition J.3 (Artifact manifest)**.** The Paper II artifact manifest is a table with rows

$$(\text{artifact name}, \text{filename}, \text{version}, \text{date}, \text{checksum}, \text{proof label}).$$

Proposition J.4 (Artifact reproducibility requirement)**.** *A downstream paper may cite a Paper II artifact as released only if the artifact appears in the manifest with filename, version, date, checksum, and proof label. If no manifest entry exists, the manuscript must describe the artifact as planned, omitted, or not part of the present release.*

Proof. A released artifact is reproducible only if a reader can identify and verify the exact file. The tuple (filename, version, date, checksum) establishes file identity, and the proof label identifies the mathematical claim it supports. □

Object	Required proof or ledger	Consumer
$S_Y\|_X \simeq S_X \otimes_{\text{gr}} S_{\text{int}}$	Spin, Spin^c, or local domain declared in Section B.	GU III, GU IV
Gamma split	$\Gamma_\mu = \gamma_\mu \otimes 1$, $\Gamma_i = \gamma^5 \otimes \kappa_i$.	GU III
Family submodule	$W^{\text{LH}}_{\text{fam}}$ inside S_{int}, selected inside the declared one-family minimal compact completion category.	GU III, GU IV
Minimal chiral completion	Family-universal abelian directions lie in span$\{Y, B-L\}$; modulo Y, $B-L$ is the anomaly-safe completion direction.	GU III
Spin(10) envelope	Minimal-rank simple chiral envelope with $\mathbf{16}\downarrow_{G_{\text{PS}}} = (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$.	GU III, GU IV
Family projector	$[\Pi_{\text{fam}}, \rho_{\text{PS}}(g)] = 0$.	GU III
One-family table	All-left-handed table and multiplicity count 16.	GU III, GU IV
Anomaly cancellation	Perturbative, global, $B-L$, Pati–Salam, and I_6 ledgers.	GU III
Yukawa channels	Gauge-admissible $(1,2,2)$ and optional $(15,2,2)$.	GU III, optional GU IV
Gauge normalization	Y, g_{BL}, and hypercharge kernel relation.	GU III, GU IV
Current definitions	J^μ, J^μ_5, n_5, spin-density convention, conservation/-source class.	GU III, GU IV
σ_0 handoff	$\sigma_0^2 = C_{55} n_{5,0}^2$ only on conserved/effectively conserved branch.	GU IV
Sterile dense-fermion branch	Species-resolved source-class packet GU2-K with occupied species, weak-singlet/sterile carrier status, polarization, washout, and energy-budget gates.	GU III; GU IV by source adapter; GU V audit
Artifacts	Manifest entries with filename, version, date, checksum, proof label.	GU III, GU IV

Final Paper II export bundle.

$$\text{GU2-Final} = \{S_Y|_X \simeq S_X \otimes_{\text{gr}} S_{\text{int}},\ \Gamma_A,\ P_L,\ P_R,\ W_{\text{fam}},\ \Pi_{\text{fam}}\},$$

$$[\Pi_{\text{fam}}, \rho_{\text{PS}}(g)] = 0, \qquad Y = T^3_R + \frac{1}{2}(B-L),$$

$$\text{span}\{Y, B-L\}, \qquad \mathbf{16}_{\text{Spin}(10)}\downarrow_{G_{\text{PS}}} = (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}),$$

state table, Pati–Salam branchings, anomaly ledger, Yukawa seed ledger, gauge-normalization ledger,

$$J^\mu_5, \quad n_5, \quad s^{\mu\nu\lambda}, \quad g_{5,\text{eff}}, \quad \sigma_0^2 = C_{55} n_{5,0}^2,$$

GU2-K : {sterile dense-fermion branch source-class packet}.

Proposition J.5 (Downstream import consistency)**.** *If GU III or GU IV imports the Paper II matter ledger, then the imported object must be one of the objects listed above or else be added as a new theorem or labeled external assumption. In particular, downstream use of the sterile dense-fermion branch requires the GU2-K source-class packet, the GU III legality status, and the* *declared washout and energy-budget gates; it is not imported as a universal value of σ_0, as an observable map, or as a detector-amplitude claim.*

Proof. The appendices prove a finite list of exports. The downstream import table and final export bundle list those proved objects. Anything outside that list is not a Paper II export. □

Export J.6 (Reproducibility and downstream export)**.** Paper II exports a reproducibility requirement: downstream use of the matter ledger must cite the relevant manifest versions if artifacts are claimed as released. The manifest includes the one-family state table, anomaly ledger, Pati–Salam branching ledger, gauge-normalization ledger, current ledger, minimal chiral-completion ledger, Spin(10) envelope ledger, sterile dense-fermion branch ledger, and export manifest when those artifacts are part of the release. The downstream import set is restricted by theorem J.5.

References

[1] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin Geometry*. Princeton University Press, 1989.

[2] Mikio Nakahara. *Geometry, Topology and Physics*. 2nd edition, Institute of Physics Publishing, 2003.

[3] Jogesh C. Pati and Abdus Salam. Lepton number as the fourth color. *Physical Review D*, 10:275–289, 1974.

[4] R. Slansky. Group theory for unified model building. *Physics Reports*, 79:1–128, 1981.

[5] Ta-Pei Cheng and Ling-Fong Li. *Gauge Theory of Elementary Particle Physics*. Oxford University Press, 1984.

[6] Paul Langacker. Grand unified theories and proton decay. *Physics Reports*, 72:185–385, 1981.

[7] Rabindra N. Mohapatra and Palash B. Pal. *Massive Neutrinos in Physics and Astrophysics*. World Scientific, 1991.

[8] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. Addison-Wesley, 1995.

[9] Steven Weinberg. *The Quantum Theory of Fields, Volume II: Modern Applications*. Cambridge University Press, 1996.

[10] Reinhold A. Bertlmann. *Anomalies in Quantum Field Theory*. Oxford University Press, 1996.

[11] Adel Bilal. Lectures on anomalies. arXiv:0802.0634, 2008.

[12] Luis Alvarez-Gaume and Edward Witten. Gravitational anomalies. *Nuclear Physics B*, 234:269–330, 1984.

[13] Edward Witten. An $SU(2)$ anomaly. *Physics Letters B*, 117:324–328, 1982.

[14] Edward W. Kolb and Michael S. Turner. *The Early Universe*. Addison-Wesley, 1990.

[15] Scott Dodelson. *Modern Cosmology*. Academic Press, 2003.

[16] Viatcheslav Mukhanov. *Physical Foundations of Cosmology*. Cambridge University Press, 2005.

[17] Mikko Laine and Aleksi Vuorinen. *Basics of Thermal Field Theory*. Springer, 2016.

[18] Bernard F. Schutz. Perfect fluids in general relativity: velocity potentials and a variational principle. *Physical Review D*, 2:2762–2773, 1970.

[19] J. David Brown. Action functionals for relativistic perfect fluids. *Classical and Quantum Gravity*, 10:1579–1606, 1993.

[20] J. Cox. *Minimal Chiral Completion and the Axial Einstein–Cartan Contact: Pati–Salam,* Spin(10)*, and the Positive C_{55} Sign Corridor*, Zenodo, 2026. doi:10.5281/zenodo.20450613.